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OF

LONDON.

PROCEEDINGS.

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1911

THE PHYSICAL SOCIETY OF LONDON.

1910-11. ? 11-12

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1910--1911.

October 28th, 1910.

Meeting held at King's College by kind invitation of Prof. E. Wilson.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Papers were read :—

1. Demonstration of a New Method for Producing High-tension Discharges. By Prof. Ernest Wilson and Mr. W. H. Wilson.
2. The Behaviour of Steel under Combined Static Stress and Shock. By Mr. F. Rogers.

November 11th, 1910.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Papers were read :—

1. On the Supposed Propagation of "Equational" Magnetic Disturbances with Velocities of the Order of 100 miles per second. By Dr. C. Chree, F.R.S.
2. On Cusped Waves of Light and the Theory of the Rainbow. By Prof. W. B. Morton.

November 25th, 1910.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Papers were read :—

1. The Electric Stress at which Ionisation begins in Air. By Dr. A. Russell.

2. The Afterglow of the Electric Discharge. By Prof. R. J. Strutt, F.R.S.

A surface brightness photometer was exhibited by Mr. J. S. Dow.

A Paper on The Approximate Solution of Various Boundary Problems by surface integration combined with freehand graphs, by Mr. L. F. Richardson, was taken as read.

December 20th, 1910.

An Exhibition of Apparatus was held at the Imperial College of Science and Technology, the following firms taking part :—

R. & J. Beck ; British Radio-Telegraph and Telephone Co., Ltd. ; Cambridge Scientific Instrument Co. ; Casella & Co. ; A. C. Cossor, Ltd. ; H. W. Cox & Co. ; J. H. Dallmeyer Ltd. ; Elliott Bros. ; Everett, Edgcumbe & Co. ; Evershed & Vignoles Ltd. ; Foster Instrument Co. ; Gallenkamp & Co. ; Gambrell Bros. ; J. J. Griffin & Sons ; J. J. Hicks & Co. ; A. Hilger Ltd. ; India Rubber, Gutta Percha & Telegraph Works, Ltd. ; Isenthal & Co. ; E. Leitz ; Marconi's Wireless Telegraph Co. ; Leslie Miller ; Morris & Lister Ltd. ; Muirhead & Co. ; Negretti & Zambra ; Newton & Co. ; R. W. Paul ; W. G. Pye & Co. ; Reynolds & Bransom ; Strange & Graham, Ltd. ; H. Tinsley & Co. ; Townson & Mercer ; Weston Electrical Instrument Co. ; Wright & Co. ; Carl Zeiss, Ltd.

The following short discourses were given :—

On some Improvements in Transmitters and Receivers for Wireless Telegraphy. By Prof. J. A. Fleming, M.A., D.Sc., F.R.S.

Short Cinematograph Demonstrations of Moving Diagrams of Lines of Force, by Prof. S. P. Thompson, F.R.S., and of Sound-wave Forms, by Prof. R. W. Wood, were exhibited by Mr. R. W. Paul.

January 27th, 1911.

Meeting held at University College, Gower Street, by kind invitation of Prof. F. T. Trouton.

Prof. C. H. Lees, F.R.S., Vice-President, in the Chair.

The following Papers were read :—

1. A Demonstration of the Phase Difference between the Primary and Secondary Currents of a Transformer by means of Simple Apparatus. By Prof. F. T. Trouton, F.R.S.

2. A Note on the Experimental Measurement of the High Frequency Resistance of Wires. By Prof. J. A. Fleming, F.R.S.

3. The Measurements of Energy Losses in Condensers Traversed by High Frequency Electric Oscillations. By Prof. J. A. Fleming and Mr. G. B. Dyke.

4. Some Resonance Curves taken with Impact and Spark Ball Dischargers. By Prof. J. A. Fleming and Mr. G. B. Dyke.

Annual General Meeting.

February 10th, 1911.

Meeting held at the Imperial College of Science and Technology
Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Report of the Council was read by the Secretary :

Since the last Annual General Meeting there have been held eleven ordinary Meetings and two informal Meetings. Of these, ten were held at the Imperial College of Science, one at King's College, one at University College, and one at the National Physical Laboratory.

The average attendance, excluding the Exhibition and the informal Meeting at the National Physical Laboratory, was 55, as compared with 41 last session.

The sixth Annual Exhibition of Apparatus by Manufacturers was held on December the 20th in the afternoon and evening and attracted a larger attendance than in any previous year, the number of Fellows and Visitors being about 800. An experimental discourse on " Some Improvements in Transmitters and Receivers for Wireless Telegraphy " was kindly given by Prof. J. A. Fleming, D.Sc., F.R.S., at both Sessions, and Mr. R. W. Paul gave demonstrations of some magnetic and optical phenomena by means of cinematograph films. The number of exhibitors was 35.

The past session has been marked by an important change in the publications. It has been felt by the Council for some time that the delay experienced in publishing the " Proceedings " hitherto has been harmful to the Society, and they have therefore decided henceforth to issue the "Proceedings" in five parts, at two-monthly intervals, during each Session, and they hope it may be possible later on to issue ten parts at monthly intervals.

This change has involved the termination of the arrangement which has existed hitherto with Messrs. Taylor & Francis, whereby Papers selected by the Council have been offered to the Editors of the "Philosophical Magazine," and have in most cases been published in that journal. Papers now read before the Society, generally speaking, only appear in the Society's "Proceedings"; but, after reading, Fellows are at liberty to offer their Papers to any scientific publication, and revised proofs are in any case sent by the Council to a selected number of journals in the hope of promoting wider publication.

The present arrangement for a two-monthly issue enables the Society to publish accepted Papers very promptly. The first part of the current volume was issued last month after some delay due to the change in arrangements, and the second part will be issued in the next few days.

The Council hope that the improvement in the rapidity of publication will lead to an increase in the membership of the Society. Although the new scheme does not involve more expense, it is important that the membership should be increased as much as possible in order to provide the funds necessary to enable the Council to widen the scope and increase the activity of the Society. The Council have appointed Mr. A. Eagle as Assistant Secretary.

The number of Ordinary Fellows now on the roll, as distinct from Honorary Fellows, is 433, a decrease of 3 on the number last year; 12 new Fellows have been elected and one Fellow who resigned a few years ago has been reinstated. Three Honorary Fellows were elected at the last Annual General Meeting, namely, Prof. S. Arrhenius, Madame S. Curie and Prof. G. E. Hale. There have been 7 resignations, and 9 Fellows have been struck off the Register either for non-payment of subscriptions or because they cannot be traced. The Society has to mourn the loss by death of Major P. Cardew, Mr. H. W. Eve and Sir William Huggins, K.C.B.

The Report of the Council was adopted.

The Report of the Treasurer was read by the Treasurer.

The total income of the Society for the year 1910 is £19. 13s. 3d. less than the previous year. The difference is mainly due to a falling-off in the subscriptions, and this can be accounted for by arrears, which amounted at the end of the year to £44. 2s., as against £20 in the last accounts. If allowance is made for this difference it will be seen that the subscriptions to the Society are nearly stationary instead of increasing as they should do. A favourable point in the Society's balance-sheet is the increase in the revenue from the sales of publications. It is to be hoped that the new arrangement for more prompt publication of the "Proceedings" will still further increase this item.

Turning to the expenditure side of the accounts the sum spent upon the publications of the Society is still unduly high. This is mainly due to the commendable efforts of our Secretary to catch up the arrears which had accumulated in past years. The arrears are completely worked off as far as the "Proceedings" of the Society are concerned, but some of the expenditure in connection therewith will come into the accounts for the current year, and consequently make them above the normal. The total expenditure of the Society shows an increase of £21. 5s. 3d. over the previous year. The net result of the year is a deficit of £18. 15s. 5d. This unsatisfactory result again emphasises the necessity of an increase in the membership of the Society so as to give a margin of income over expenditure. The fact that the accounts show a deficit for the year 1910 need not alarm the Fellows of the Society, as it is mainly caused by the large sums the Society has been spending on its printing account.

The present accounts show for the first time a separate statement of the life compositions received by the Society, as the Council has decided that during the life of any Fellow who has compounded for his subscription all sums paid by him as composition shall be invested, and the interest thereon only shall be applied to the current expenditure of the Society, and after his death the amount of the composition shall be transferred to the general funds of the Society.

In order to start the fund the compositions paid by each of the life Fellows of the Society on the books at the 31st December last have been included in a separate account, the total of which amounts to £2,209. 10s.

In future all composition fees will be credited to this account, and on the death of a life Fellow his composition will be transferred to the current account.

The property account of the Society shows but little change. What change there is is caused by the above-mentioned deficit on current account, and by market fluctuations of the securities in which the Society's funds are invested. The figures appearing in the account are the value of the securities at the market prices on December 31st, 1910, kindly supplied by the Manager of Parr's Bank.

Following the custom of previous years the publications in stock appear in the property account at £200. They have not been revalued. During the current year the publications in stock will be revalued, and the new valuation will appear in the current year's accounts.

The Report of the Treasurer and the Balance-Sheet, which was also presented to the Meeting, were adopted.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY, 1910, TO DECEMBER 31ST, 1910.

Dr	£	*s.*	*d.*	£	*s.*	*d.*
Entrance Fees	16	16	0			
Subscriptions	457	16	0			
Extra Subscriptions	28	5	0			
Arrears	42	0	0			
Abstracts, &c.	10	16	0			
Students	3	13	6			
				559	6	6
Composition Fees				21	0	0
Dividends :—						
Furness Debenture Stock	15	1	2			
Midland Railway	37	13	4			
Metropolitan Board of Works	6	12	0			
Lancaster Corporation Stock	11	6	0			
New South Wales	8	7	6			
London, Brighton & South Coast Railway	24	2	7			
Great Eastern Railway	18	16	8			
				121	19	3
Sales of Publications				55	1	11
One Sale of Royal Society Proceedings				1	1	0
Income Tax recovered				26	12	8
Interest on £400 deposit account				8	17	10
				793	19	2
January 1, 1910, balance at Bank				78	9	7
				£872	8	9

Cr.	£	*s.*	*d.*	£	*s.*	*d.*
"Science Abstracts"				277	10	0
Taylor & Francis :—						
Proceedings	250	12	9			
Abstracts, &c.	104	13	9			
Distribution	32	0	6			
Separate Copies	36	18	6			
				424	5	6
Periodicals				6	6	3
Binding				2	8	6
Reporting				35	0	0
Attendance				13	12	6
Refreshments and sundry expenses at meetings				12	18	4
Petty cash				19	10	6
Bank commission & stamped cheques				0	6	7
Subscriptions paid in error				2	7	0
Royal Astronomical Society				2	2	0
Royal Society				1	1	0
Deficit on Dinner Account				3	14	1
International Catalogue (2 years)				11	12	4
				812	14	7
Balance at Bank	164	8	0			
Less two Cheques	104	13	10			
	59	14	2	59	14	2
				£872	8	9

WILLIAM DUDDELL, *Honorary Treasurer.*

Audited and found correct,

C. W. S. CRAWLEY.
A. WRIGHT.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31ST, 1910.

ASSETS.	£	s.	d.
Subscriptions due, Treasurer's estimate...	44	2	0
£533 Furness Ry. Co.3 per cent. Debenture Stock	405	0	0
£1,600 Midland Railway 2½ per cent. Preference Stock	1,064	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	200	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	324	0	0
£284. 2s. 9d. New South Wales 3½ per cent. Inscribed Stock	246	0	0
£500 London, Brighton & South Coast Railway Ordinary Stock	565	0	0
£500 Great Eastern Railway 4 per cent. Debenture Stock	525	0	0
Balance at Bank	164	8	0
Ditto on deposit........	400	0	0
Publications in stock	200	0	0
	£4,137	10	0

LIABILITIES.	£	s.	d.
Two Cheques not presented	104	13	10
Life Composition Fund	2,209	10	0
Balance	1,823	6	2
	£4,137	10	0

WILLIAM DUDDELL, *Honorary Treasurer.*

Audited and found correct,

C. W. S. CRAWLEY.
A. WRIGHT.

LIFE COMPOSITION FUND.

		£	s.	d.
186	Fellows paid £10	1,860	0	0
3	Fellows paid £15	45	0	0
4	Fellows paid £21	84	0	0
7	Fellows paid £31. 10s.	220	10	0
200		£2,209	10	0

Audited and found correct,

WILLIAM DUDDELL, *Honorary Treasurer.*

C. W. S. CRAWLEY
A. WRIGHT.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows :—

President.—Prof. H. L. CALLENDAR, M.A., F.R.S.

Vice-Presidents who have filled the Office of President.—Prof. G. C. FOSTER, F.R.S., Prof. W. G. ADAMS, M.A., F.R.S. ; Prof. R. B. CLIFTON, M.A., F.R.S. ; Prof. A. W. REINOLD, M.A., F.R.S. ; Prof. Sir ARTHUR W. RÜCKER, M.A., D.Sc., F.R.S. ; Sir W. DE W. ABNEY, R.E., K.C.B., D.C.L., F.R.S. ; Prin. Sir OLIVER J. LODGE, D.Sc., F.R.S. ; Prof. S. P. THOMPSON, D.Sc., F.R.S. ; R. T. GLAZÉBROOK, D.Sc., F.R.S. ; Prof. J. H. POYNTING, M.A., F.R.S. ; Prof. J. PERRY, F.R.S. ; C. CHREE, Sc.D., F.R.S.

Vice-Presidents.—A. CAMPBELL, B.A. ; Prof. C. H. LEES, D.Sc., F.R.S. ; Prof. T. MATHER, F.R.S. ; S. SKINNER, M.A.

Secretaries.—W. R. COOPER, M.A. ; S. W. J. SMITH, M.A., D.Sc.

Foreign Secretary.—Prof. S. P. THOMPSON, D.Sc., F.R.S.

Treasurer.—W. DUDDELL, F.R.S.

Librarian.—S. W. J. SMITH, M.A., D.Sc.

Other Members of Council.—W. H. ECCLES, D.Sc.; A. GRIFFITHS, D.Sc. ; Major W. A. J. O'MEARA, C.M.G. ; A. RUSSELL, M.A., D.Sc. ; W. N. SHAW, M.A., Sc.D., F.R.S. ; F. E. SMITH ; Prof. the Hon. R. J. STRUTT, F.R.S. ; W. E. SUMPNER, D.Sc. ; R. S. WHIPPLE ; R. S. WILLOWS, M.A., D.Sc.

Prof. J. H. VAN'T HOFF was elected an Honorary Fellow of the Society.

Votes of thanks were passed to the Auditors, to the Officers and Council, and to the Governors of the Imperial College of Science and Technology.

Prof. H. L. CALLENDAR then delivered the President's Address.

PRESIDENTIAL ADDRESS.

The Society has been exceptionally fortunate this year in losing so few of its Members by death. So far as I know, there are only three Members to commemorate. The most important part of their work lay outside the scope of the activities of this Society, and I feel that any attempt which I may make to do honour to their memory will of necessity be inadequate, and that their achievements will be more fittingly recorded elsewhere.

The great pioneer of stellar spectroscopy, Sir W. HUGGINS, was one of the earliest members of our Society, and rendered good service as Member of Council and Vice-President. His duties as President of the Royal Society, and in other important capacities, prevented his frequent attendance at our Meetings during recent years, but many Members of our Society have profited by his inspiration, and will hold him in grateful remembrance for his kindly advice and assistance.

Mr. H. W. EVE joined the Society during its first Session in 1874, and did much to develop the teaching of science at University College School, where some of our Members received their first introduction to physics.

Major P. CARDEW, R.E., who joined the Society in 1896, as head of the electrical department of the Board of Trade was closely concerned with the establishment and legal definition of the electrical standards, which are of interest to all physicists, but his principal work as an electrical engineer lay somewhat outside the scope of our Society.

Here follows the special subject of the Address which will be found on pp. 153–189 of the "Proceedings."

February 24th, 1911.

Meeting held at the Imperial College of Science and Technology

Prof. H. L. CALLENDAR, F.R.S., President, in the Chair.

The following Papers were read :—

1. On Flames of Low Temperature Supported by Ozone. By Prof. R. J. STRUTT, F.R.S.

2. On The Movement of a Coloured Index along a Capillary Tube and its Application to the Measurements of the Circulation of Water in a Closed Circuit, was read by Dr. ALBERT GRIFFITHS.

3. An Optical Lever of High Power suitable for the Determination of Small Thicknesses and Displacements was exhibited by Mr. E. H. RAYNER.

March 10th, 1911.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. CALLENDAR, F.R.S., President, in the Chair.

A Demonstration of the Working of the Anschütz Gyro Compass, illustrated by lantern slides and models, was given by Mr. G. K. B. ELPHINSTONE.

A Paper by Dr. W. H. Eccles, entitled Note on an Electrical Trevelyan Rocker, and one by Dr. G. W. C. Kaye, entitled Some Notes on the Tilted Gold Leaf Electrometer, were taken as read, the apparatus in each case being on the table for inspection afterwards. A Paper entitled The Roots of the Neumann and Bessel Functions, by Mr. J. R. Airey, was also presented to the meeting and taken as read.

March 24th, 1911.

Meeting held at the City and Guilds Engineering College [by kind invitation of Prof. T. Mather, F.R.S.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

1. A Sensitive Thermo-Regulator was shown by Dr. H. F. Haworth.
2. Some Experiments in the Measurement of Electrolytic Resistance using Alternating Currents were shown by Dr. H. F. Haworth.
3. A Paper on Oscillatory Currents in Coupled Circuits was read by Prof. G. W. O. Howe.
4. Some Radio-telegraphic Apparatus in Use at the City and Guilds College was briefly exhibited by Prof. G. W. O. Howe.

April 28th, 1911.

Meeting held at the Imperial College of Science and Technology.
Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Papers were read :—

1. On High Tension Electrostatic Wattmeters. By Prof. E. Wilson.
2. On the Behaviour of Incandescent Lime Cathodes. By Dr. R. S. Willows and Mr. T. Picton.
3. On the formation of Dust Striations by an Electric Spark. By Dr. S. Marsh and Mr. W. H. Nottage.

A Paper by Prof. E. Wilson and Mr. L. C. Budd on Previous Magnetic History as affected by Temperature was taken as read.

May 12th, 1911.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

1. Some Diagrams of Stream Lines past an Elliptic Cylinder were exhibited by Sir George Greenhill and Col. R. L. Hippisley.

2. A Paper on The Method of Constant Rate of Change of Flux as a Standard for determining Magnetisation Curves of Iron, by J. T. Morris and T. H. Langford, was read by Prof. J. T. Morris.

3. An Electric Thermo Regulator was exhibited by the President.

May 26th, 1911.

An informal Meeting was held at the National Physical Laboratory, where a number of demonstrations were given and Fellows had an opportunity for inspecting the laboratories.

June 9th, 1911.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Papers were read :—

1. The Lüders' Lines on Mild Steel. By Mr. W. Mason.

2. A New Method of Harmonic Analysis by Averaging Selected Ordinates. By Prof. S. P. Thompson.

The following Demonstrations were given :—

1. A Demonstration of the Subjective Nature of the Difference Tone. By Prof. S. P. Thompson.

2. A Demonstration of Spinning Tops and Gyroscopic Apparatus. By Sir George Greenhill.

3. A Model Illustrating the Passage of a Light Wave through Quartz was described by Dr. H. S. Allen.

The following Papers were taken as read :—

1. The Measurement of Contact Differences of Potential. By Prof. A. Anderson and Mr. J. E. Bowen.

2. A Short Table of Circular and Hyperbolic Functions for Complex Values of the Argument. By Mr. A. Johnstone.

June 30th, 1911.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

1. A Paper on the Capacity Coefficients of Spherical Electrodes was read by Dr. A. Russell.

2. An Exhibition of the Benkö Primary Battery was given by Mr. W. R. Cooper.

A Paper on the Effect of a Narrow Saw-cut in the Edge of a Conducting Strip on the Stream Lines in the Strip and on the Resistance of the Strip, by Prof. C. H. Lees, F.R.S., was taken as read.

XXIX. *Some Radio-telegraphic Apparatus in Use at the City and Guilds Engineering College. By Prof.* G. W. O. Howe.

Read March 24, 1911.

Although experiments in wireless telegraphy have formed part of the third-year course in electrical engineering for the last four years, it is only recently that attempts have been made to pick up signals from a considerable distance. By means of a new antenna of great height, and the receiving apparatus which is fitted up on the lecture table this evening we have succeeded in picking up the signals sent out from the Marconi station at Glace Bay, Nova Scotia.

The antenna employed for this work consists of a single wire of No. 14 copper-coated steel. The upper end is attached to the central tower of the Imperial Institute, 260 ft. above the ground. It is brought across in a single span of 530 ft. to a chimney stack on the college, from whence a wire passes down into the laboratory. For the purposes of the demonstration another wire has been connected to the antenna and brought into the lecture theatre. The fundamental wave-length of this antenna is about 900 metres.

In addition to this a T antenna is suspended horizontally above the roof of the college. It consists of five parallel wires 190 ft. long and 110 ft. high. A single wire connects the mid point to the laboratory. The fundamental wave-length is about 520 metres.

The receiving transformer or "jigger" used for short wave-lengths is on the principle of the Ayrton-Perry variable standard of self-induction, the primary coil being rotated within the fixed secondary in order to vary the coupling.

The wave-length of the electromagnetic waves sent out by the Marconi stations at Clifden, on the west coast of Ireland, and at Glace Bay is over 6 km. In the special transformer constructed for receiving these low-frequency waves, the coils are square, being supported at the corners on ebonite strips fitted to a wooden frame. The primary coil is movable, and can be raised 4 ft. above the fixed secondary by turning a handle. A lead weight below the transformer serves to counterbalance the weight of the moving coil. Each coil is subdivided into three parts, and the ends brought to a plug board. The wire consists of 3×3×3—*i.e.*, 27 wires No. 36, each wire single covered, and the whole double silk covered over all. This ensures the high-frequency currents being distributed uniformly between the 27 wires. The primary has 160 turns 13·5 in. square, the winding length is 16 in., and the inductance of the coil is 6·45 millihenrys. The fixed secondary has 187 turns 15 in. square, the winding length is 13 in., and the inductance 12·9 millihenrys. The height over all when the coupling is a minimum is about 9 ft.

If the inductance, which must be placed in series with the aerial to tune it to various wave-lengths, be plotted against the square of the wave-length, the curve obtained is a straight line through the origin, except for short wave-lengths, where the inductance of the aerial itself is comparable with that placed in series with it. The slope of the line depends only on the capacity of the aerial, which, in the present case, is 1·35 milli-microfarads, so that an inductance of about 7·8 millihenrys must be placed in series with the aerial to tune it up to a wave-length of 6,200 metres. The exact inductance required for any wave-length is obtained by using suitable coils of the transformer primary and by adjusting the so-called tuning coils; but for shorter wave-lengths than 1,200 metres, capacity must be inserted in the aerial tuning condenser.

The secondary circuit consists of a part or all of the secondary coil of the transformer and a variable condenser. As arranged the apparatus has a range of 300 to 7,000 metres wave-length.

By calibrating the secondary oscillating circuit it can be used as a wave meter in determining the wave-length of the signals received. There are several sources of error, however, to be taken into account. The coupling must be sufficiently loose, as the tuning in the secondary circuit changes with the coupling when the latter is too tight. Another source of error is the detector, which in the arrangement used is shunted across the condenser. Switching over from one detector to another will sometimes cause an apparent change in the frequency of the received signal. If the variable capacity be small, an error is introduced if the self-capacity of the coil itself be neglected. If the coil acted as a simple inductance, and had no self-capacity, a curve connecting wave-length and $\sqrt{K}$ would be a straight line through the origin; but even when the capacity K of the variable condenser is reduced to zero the coil itself has a natural frequency of its own, so that for $K=0$ the wave-length is not zero, but has a perfectly definite value. When K is large the error introduced by neglecting this effect is negligibly small, but when K is small the error becomes very important. It is not only the effective capacity which is affected, but, owing to the current being less at the ends than at the middle of the coil, the effective inductance is less than the real inductance of the coil. This subject was investigated by Drude in 1902; he established empirical formulæ for calculating the natural frequency of coils of various shapes, but there is still room for further work on the subject.

The connections of the complete receiving apparatus are shown in Fig. 1. There are two alternative detectors; those actually used for the demonstration are exactly similar, consisting of a light contact between small pieces of zincite and copper pyrites. This is very sensitive, but is rather troublesome, as it needs frequent adjustment as a rule, and is apt to be rendered inoperative by a bad "atmosphere." The Glace Bay signals can only be heard at night, and the necessary coupling is rather tight; this makes the atmospheric distarbances very troublesome, it being sometimes impossible to hear the signals through them for several seconds. Darkness favours the transmission of electromagnetic waves,

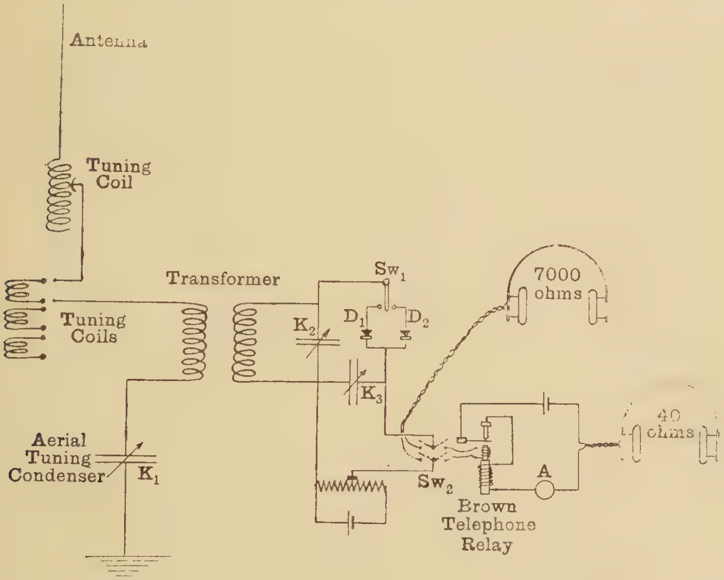

FIG. 1.

irrespective of their source, and, as lightning flashes are probably much more powerful than the waves sent out by a radio-telegraph station, one may very well hear every lightning flash on the globe as a crack in the telephone. The disturbances often resemble the rattle of musketry. This is undoubtedly the greatest drawback to long-distance radio-telegraphy.

It is evident that the most sensitive detector is not necessarily the most advantageous. One must sacrifice sensibility to obtain reliability. The relative sensibility and convenience of different types of detectors is a matter on which one should hesitate before giving any dogmatic opinion, on account of the different conditions under which each type gives its best results.

The detectors used at the college are on the lecture table ; they include the filings coherer with inker, &c. (the standard apparatus five or six years ago), the electrolytic with various sizes of wire, the magnetic detector and the Fleming valve. The minerals in the detectors used in the demonstration can be replaced by others in a few seconds.

The telephone head set used with the zinc-pyrites detector has 7,000 ohms resistance. By means of a switch this telephone can be cut out and its place taken by a Brown telephone relay* operating a 40-ohm head set, in which the ordinary earpieces have been replaced by trumpets. With this arrangement the messages and time signals sent out from the Eiffel Tower and from Norddeich, near Wilhelmshaven, can be distinctly heard by every one in the large lecture theatre.

It is only by using the telephone relay that the signals from Canada can be heard ; unfortunately, the relay does not manifest a wise discrimination between radio-telegrams and atmospheric disturbances. This could probably be improved by tuning the reed to the pitch of the musical note of the sending apparatus—that is, to the spark frequency.

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[*] *See* "Trans." of Inst. of Elec. Engineers, Vol. XLV., p. 590.

XXX. *The Lüders' Lines on Mild Steel. By* W. Mason, *M.Sc., The University, Liverpool.*

First Received March 29, 1911. Received in Revised Form May 26, 1911.

At the time of elastic breakdown under stress of all the ductile metals, slipping occurs along surfaces of weakness in the crystals. Such slips are apparent as "slip-lines" or "slip-bands," when microscopic examination is made of surfaces of the specimens which have been polished and etched before straining.* Moreover, on the surface of iron and steel specimens there become visible, at or about the yield point, lines or bands, known as "Lüders'" or "Hartmann's" lines.† These lines exhibit themselves on the large scale, commonly extending over a large extent of the surface of the strained specimen. Somewhat similar lines have been observed by Hartmann and others on the surfaces of many other ductile metals, but there seems to be some doubt whether these are of the same kind as those on iron and soft steel.‡ The writer has confined his attention to the last-mentioned material, and it has been his object to collect additional information concerning the occurrence of these lines upon mild steel, to determine the angular incidence of these lines, and also, if possible, the angles of the corresponding internal surfaces with respect to the stress directions, in specimens loaded in various ways.

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* Ewing and Rosenhain, "Phil, Trans" A, 1900, Vol. 195.

† Hartmann. "Distribution des déformations dans les métaux soumis a des efforts," 1896, Paris, Berger-Levrault et Cie.

G. H. Gulliver. "Proc." Inst. Mech. Engineers, No. 1, 1905.

G. H. Gulliver. "Proc." Roy. Soc., Edinburgh, Vol. XXX., Part I.

G. H. Gulliver. Congrès International des Méthodes d'Essai, 1909.

Fremont. "Bulletin" de la Société d'Encouragement, Sept. 1903; or "Nature," Jan. 21, 1904.

Also *see* Congrès International des Méthodes d'Essai, 1900, I., 95; 1900, I., 143; 1900, I., 185.

‡ G. H. Gulliver. International Association for Testing Materials, 1909, VIII., 9, pages 2 and 3 of translation into English. "From the results of his (Mr. Gulliver's) experiments on most of the metals and alloys on which Hartmann and others claim to have observed the lines of Lüders', Mr. Gulliver is of opinion that what these investigators really obtained were contractile depressions. So far as Mr. Gulliver's experience goes, the lines of Lüders' are only found on iron and steel—metals which have a well-marked yield point." The author has recently tested some drawn lead tubes, of various wall thicknesses, under end pressure and also under simultaneous end pressure and internal fluid pressure. No lines were observed. Lead is one of the metals on which the lines were said to have been obtained. The lead tubes were in the author's possession for four months before being tested—an item not unimportant in view of the phenomenon of recrystalization in lead observed by Ewing and Humphrey.

On mild steel specimens in tension these lines occur on angles not differing much from 50 deg. to the direction of pull; and although, so far as the writer is aware, no definite information up to the present is obtainable concerning the direction of the corresponding internal surfaces, it seems probable that the latter are inclined at the same angle, viz., about 50 deg.

It is remarkable that these lines should be so regular and uniform in angle on iron and steel specimens in tension. They are no doubt mainly due, like the crystalline cleavage slips of Ewing and Rosenhain, to shear stress. In a tension specimen, however, the planes of maximum shear stress are any ones of a family all at 45 deg. to the direction of pull; and, moreover, the orientation of the crystals, from grain to grain, is entirely random. Under such conditions, it would not be expected that any continuous series of grains in which slipping has occurred in the same direction would be possible, much less any continuous surfaces or lines of slipping. The phenomenon is one very intimately related to that of the yield point, the latter, if not confined entirely to iron and steel, being absent or only partially exhibited by other metals.

On mild steel specimens in compression the Lüders' lines do not appear, from the published work, to be so regular in inclination as for tension. Mr. G. H. Gulliver finds that the angle lies usually between 40 deg. and 45 deg. and sometimes "even above 45 deg." to the direction of the compression.* The writer finds, in tube specimens of mild steel, that the angle is always greater than 45 deg., and is about 50 deg., the same as for tension. (*See* Tables A, B, C, D.) (The lines at 90 deg. are apparent exceptions only; these are unmistakably traces of conical surfaces—the axis of the cone being coincident with the axis of the tube.) He has found, too, when tensile and compressive stresses of equal intensity are applied simultaneously in directions at right angles to each other that the lines are at 45 deg. to these directions, or thereabouts.†

Occurring as these lines do at or about the yield point of the mild steel, a study of them is likely to throw light on the connection between the process of yielding and the stresses causing it.* The writer decided to continue to use tubular specimens because of the difficulty of applying (sensibly) uniform stresses, except by their use, in two directions at right angles. It was easy, too, to subject a tube of fairly large diameter and small thickness to longitudinal compression as well as to a small amount of hoop tension, by internal fluid pressure; and by so doing to give the planes of maximum shear stress two determinate directions only. This done, it seemed probable that the directions of slipping would be more regular and uniform than they are when longitudinal compression only is applied, and when, therefore, *all* planes at 45 deg. to the axis of the tube are planes of maximum shear.

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* "Proc." Roy. Soc., Edinburgh, Vol. XXIX., Part V., No. 28, page 439.
† "Proc." Inst. Mech. Eng., Vol. IV., 1909.

The use of tubes, too, facilitated the reversal of the stresses, *i.e.*, the making of tests with the tubes in longitudinal tension and hoop compression, as well as with end pressure and hoop tension.

Scheme of the Tests.

Four series of tests were made; three series (Tables A, B and C) were made on mild steel tubes subjected to longitudinal compressive stress simultaneously with the application of interior hydrostatic pressure; and one series (Table D), in which some of the specimens were stressed as just mentioned, while others were made to undergo longitudinal tension simultaneously with water pressure applied externally.

In each series the loads and the water pressure were applied in such proportions that a set of tests was obtained with various ratios of longitudinal stress (p_1) to hoop stress (p_2).

The stress at the point of yielding of each specimen was observed, and the angle of inclination of the spiral lines of Lüders measured.

The Test Specimens.

Those of Tables A, B and C were cut from solid drawn steel tubing of three sizes, viz.,

14 L.S.W.G. (about 0·08 in. thickness), bore 3 in.
11 ,, (about 0·118 in. ,,), bore 2·5 in.
10 ,, (about 0·127 in. ,,), bore 2·75 in.

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* Lüders' lines are not frequently observed in the usual course of testing. The writer has tested, at various times, several hundreds of bars and, excepting the specimens of this work, has observed the lines on very few of them. The most favourable condition for good development of them appears to be when one dimension of the specimen is small, *e.g.*, on flat bars or tubes. The lines were absent when tubes were tested thus:—

Simultaneous longitudinal tension and hoop tension of equal intensities.

Simultaneous longitudinal compression and hoop compression of equal intensities.

The 14 and 11 G tubing was cold drawn, while the 10 G was hot drawn. All the specimens (except those stated otherwise) were well annealed; and the ends were faced up in the lathe so as to be perpendicular to the length. Only perfectly straight specimens were tested.

It was impossible to obtain tubes of perfectly uniform thickness of wall. In nearly all the tubes the bore was slightly eccentric to the exterior surface; this eccentricity varied from 0 to 0·0035 in. in the 14 G tubes, and from 0 to 0·005 in. in the 10 G. The conical cap* through which the compressive load was applied was set on the tube so that the axis of the cone coincided with the centre of gravity of the metallic section of the tube, thus giving uniformity of compressive stress p_1. In no experiment was the eccentricity between the axis of the cone and the axis of the tube greater than 0·04 in. The hoop tensile stress, p_2, was calculated for the locality of least thickness. All the tubes were accurately calipered.

The specimens of Series D were tubes turned and bored out of hot drawn mild steel tube of bore $2\frac{3}{4}$ in. and thickness $\frac{1}{4}$ in. The finished bore was 2·86 in., and the thicknesses about 0·10 in. and 0·09 in. The thicknesses were accurately calipered. The specimens were placed in a sheath and heated in an electric furnace to a temperature 1,400°F. to 1,430°F. They were brought up to that temperature and allowed to remain thereat for 15 minutes; then they were quickly removed and allowed to cool in the sheath, the latter being placed in a covered drain pipe. The total time in the furnace was 45 to 50 minutes. The heat was controlled by a resistance coil which regulated the current, and a record was taken of the temperature of each specimen (or rather of the sheath) by a pyrometer. The main object of the treatment was not to anneal the tube, but to cover the bright turned surfaces with a thin film of scale; at the same time it was possible to treat all the specimens alike.

On all specimens the Lüders' lines were manifested by the disturbance of the scale. That the scale served as an indicator of the disturbance beneath itself, and that the lines were not merely due to the yielding of the scale under its own stressing, was evident from the extensometer indications. Evidence of this was apparent on the specimens themselves.

Apparatus for Longitudinal Compression and Hoop Tension.

The compressive loads were applied in a 100-ton Buckton vertical testing machine. In order to obtain axial loading, the load was applied through a conical cap, as described elsewhere.* Water was pumped into the interior of the tube through a hole in the base-block on which the tube sat; in all cases, except in certain of the specimens of Table D, water-tightness was obtained by thin load rings inserted between the ends of the tubes and the cone and base-block.

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* *See* Fig. 3, Plate 30, "Proc." Inst. Mech. Eng., 1909. No. 4.

When the pressure of water inside the tube necessary to obtain a given hoop tension was large, and when simultaneously with it a small longitudinal pressure only was required, the lead joint did not prove effective. Water-tightness was then secured by cup-leathers at the ends of the specimen.

In several of the tests of Series D, cup-leathers at the ends as well as lead rings were used in order that nearly the whole of the hoop tensile stress might first be imposed with no end pressure and therefore no frictional grip on the ends. The proper load for the required longitudinal compressive stress was then imposed.

The net load compressing a tube was found by subtracting from the load on the testing machine the upward thrust due to the water pressure (if any) on the area of the bore of the tube. The hoop tensile stress due to the water pressure was calculated regarding the tube as a thick cylinder, the stress p_2 in the Tables being that at the inside skin of the tube. A Ewing extensometer was used to observe the strain. It was not fixed directly by its own screw points to the tubes, which were too large in diameter for the instrument, but was fixed by means of a special detachment described elsewhere.*

Apparatus for External Pressure.

The external pressure, giving hoop compressive stress, was applied to the tubes by means of the jacket represented in Fig. 1. The tube to be tested passed through the cast-iron jacket J, whose ends are fitted with brass annular caps C_1, C_2 surrounding the tube. C_2 fits the tube loosely, while the open-in C_1 is large enough to admit the sleeve S, and to leave a space of about 1/50 in. between S and itself. The surfaces of S and C_1, forming the boundary of one end of the jacket space, are curved to fit a cup-leather L, which makes a water-tight joint between the tube and S, and between S and C_1.

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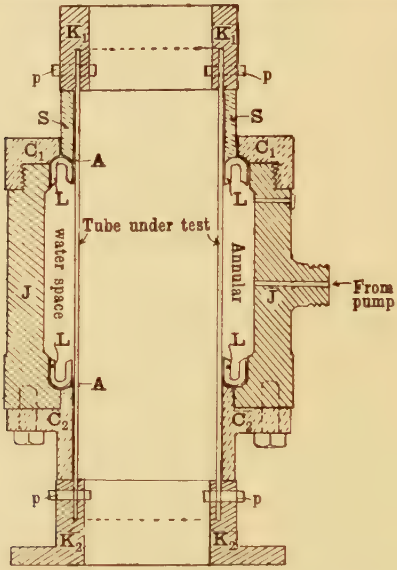
* "Proc." Inst. Mech. Eng., 1909, No. 4.

The other end of the jacket is made tight by another cup-leather L. The sleeve S and the lower end of C_2 bear against the holders K_1, K_2 which are fastened to the ends of the tube by pins *p*, *p*, which are equally spaced round the circumference. Under the pressure in the jacket, the sleeve S and C_2 (to which the main body of the jacket is bolted) tend to separate under a thrust which, being transmitted to the holders K_1, K_2, puts the tube specimen into tension. The amount of this tensile load has been taken to be: Water pressure × (sectional area of sleeve + ½ annular area between S and C_1). This amount is a little doubtful on account of the rigidity of the leather covering the 1·50 in. annular space between S and C_1, and has been marked as uncertain in Table D.



Scale: ½ Full Size.

FIG. 1.—JACKET USED FOR THE APPLICATION OF EXTERNAL PRESSURE TO THE TUBES.

Showing the arrangement used for the tests of Specimens Nos. 5, 6, 7 of Table D. Specimen No. 4 of Table D (4¾ in. long) was placed in position with its ends at the points A.A.

In the test No. 4, Table D, the tube specimen only extended to the commencement of the curve of the cup-leathers, and separation between the jacket and S was prevented by large washers kept in position by a bolt passing axially along the interior of the tube.

Gauges and their Calibration.

The gauge used to indicate the water pressure was one of the Bourdon pattern. It was calibrated up to 400 lb. per sq. in. against dead weights. From this pressure upwards calibration was performed by means of an apparatus which, set up in the testing machine, weighed the hydrostatic pressure against the load on the machine. This apparatus was checked by a standard Bourdon gauge reading up to 3,000 lb. per sq. in., which was tested at the maker's works against dead weights.

The testing machine was calibrated by the Board of Trade about the time of the experiments.

Tests in Longitudinal Compression.

All tests in compression made on solid specimens whose length is greater than two or three times the smallest lateral dimensions are subject to the difficulty of instability, with consequent bending. In the case of tubes, this bending may be "column bending" after the manner of long solid struts, or may be bending or buckling of the tube walls. The latter manner of failure results in a sort of wave formation,* and the tube finally crushes down by the folding together of the crests and hollows of the waves. When the walls are very thin the wave formation will develop early in the test, and will cause failure before the elastic limit or yield point of the material in compression is reached.

This manner of failure has been investigated by Mr. A. Mallock,† who has calculated the inclination of the salient angles of the incipient spiral folds. These lines of spiral folding do not seem to have any direct relation to the Lüders' lines. They do not agree in inclination with the latter. And the lines on the tubes do not seem, from their appearance, to be caused by bending; many of the lines and bands are very precise and cleanly marked; also in the long lines there is nothing to indicate any periodic change along the length such as would be expected if they were lines of incipient folding. Quite as frequently as otherwise the lines were spirals of one hand only. There was, too, always a difference of position, lengthwise to the tube, between a "pair" of lines running circumferentially (*see* page 316), this difference between the inside line and the outside line of the "pair" being about equal to the wall thickness; this appears to be inconsistent with the assignment of folding as a cause There is, in fact, the closest resemblance between the lines on the tubes and the Lüders' lines on flat bars strained in tension.

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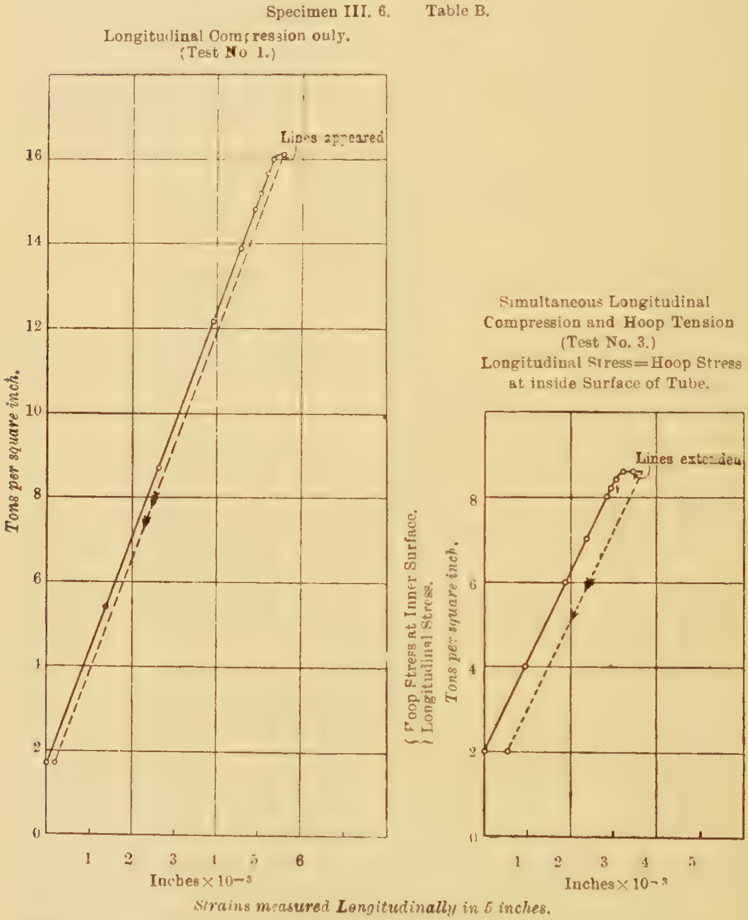
* "Secondary Flexure of Tubes." W. E. Lilley, "Engineering," Jan. 10, 1908.

† "Note on the Instability of Tubes subjected to End Pressure." "Proc." Roy. Soc., A., Vol. 81, No. 549. A. Mallock, F.R.S.



NOTE.—A curve test for No. 2 is not plotted, since the ratio between longitudinal stress and hoop stress was not kept constant during the tests.

FIG. 2A. FIG. 2B.

STRESS STRAIN DIAGRAM ILLUSTRATING THE TIME OF APPEARANCE OF THE LÜDERS' LINES.

Fig. 2A is a typical stress-strain curve; it illustrates the sharp bend at the yield point which most of the compression specimens exhibit. After a test up to the yield, the wall of a tube showed slight disturbance; but, except in one or two cases, no trace of flexure was observable before the yield and the appearance of the lines. An increase of about 3 and 8 tons per sq.in. was required to produce collapse, by the wave formation, in the 14 G and 10 G tubes respectively. With the ratio of thickness to diameter of the tubing used in the experiments, it seems probable that the yield point of the steel was reached before the commencement of flexure of the walls; and, at any rate, before any such flexure had material effect in causing failure.

Tests in Simultaneous Longitudinal Compression and Hoop Tension.

The yield point was sudden and very well defined.

Fig. 2B is a typical stress-strain curve. The increments of load and of internal pressure were arranged so as to give equal increments of compressive stress and equal increments of hoop tensile stress. At intervals during the increase of stress of the specimens of Tables C and D, the load on the testing machine was run back while the internal pressure was kept constant, until leakage of water just commenced. This was done in order to minimise frictional gripping of the tube ends. The length on which the strains were measured did not include about 1½ in. at each end of the specimen. No doubt there would be some gripping of the tube ends by the cap and pedestal and consequently some prevention of the lateral elastic expansion under the hydrostatic pressure. This effect would be diminished by the presence of the lead rings. At the compressive stresses corresponding to the yield points, the lead was always at its "pressure of fluidity." A set of three tubes of 14 gauge thickness, 3 in. bore, and 3, 6 and 18 in. long, when tested under such conditions that the compressive stress was equal to the hoop tensile stress, gave respectively for the shear stresses at the appearance of the lines 11·05, 11·25 and 11·10 tons per sq. in. If such stressing was continued beyond the yield point, the effect of the frictional gripping became very apparent; the tube became slightly barrel-shaped at the ends.

Tests in Hoop Compression with and without Longitudinal Tension.

The extensometer was fixed by means of its special attachment to the parts of the tube projecting beyond each end of the jacket (*see* p. 310). The extensometer was only intended to indicate the yield point, and not to give quantitative measurement of the strain. It was found, however, that the tubes collapsed before any decided yield point was indicated by the extensometer, showing that the means of measurement was not sensitive enough. The extensometer was not used on specimen No. 4 of Table D; instead, a strong inner tube about $\frac{1}{100}$ in. less than the bore of the test specimen was used as a feeling piece. When this became tight the test was stopped. This inner tube also prevented the complete collapse of the specimen. An inner tube, preventing collapse, was also used for specimens Nos. 5 and 6 of Table D.

The stresses due to the external pressure are marked as uncertain in the Table D on account of the aid given by the projecting ends (beyond the jacket) to the central part under pressure in the jacket. That this aid was considerable was evident from the high pressures necessary to cause collapse. The Lüders' lines were, however, very well defined, even on the specimens which collapsed. The planes of maximum shear were, in all the specimens, perpendicular to the plane of p_1 and p_2, and 45 deg. to a generator.

Mode of Occurrence of the Lüders' Lines.

In the experiments of this Paper, the first appearance of "creep," or "time effect," observable on the extensometer was very quickly followed, with only an interval of stress of 0·1 to 0·3 ton per square inch, by the large increment of strain indicating the yield point. The Lüders' lines appeared sometimes at the first indication of "creep" and sometimes at a stress interval of about the above amount afterwards.* When once a line or lines had appeared (usually at an end of the specimen) these extended at either constant value of the imposed loading or with this only very slightly increased. This extension was usually accompanied by the broadening of the part of the line which first appeared. In nearly all cases the broadening took place equally on each side of the line, and in those cases where the spreading was unequal on the two sides, the inequality appeared to be due to the presence of other adjacent lines. When a line or lines had extended so far that the slope could be well measured, the test was, as a rule, stopped. In all cases the loading and water pressure were increased very slowly when the neighbourhood of the yield was reached; but occasionally the spreading and broadening was so rapid that the whole, or nearly the whole of the surface was so much affected that the inclinations of the lines could not be measured. While the name "lines" has been used—the name commonly given to them being "Lüders'" or "Hartmann's lines"—it would be more accurate to describe many of them as bands. In some cases the original line can be very clearly seen in the centre of a broadened band. On some specimens the lines formed one system of parallel or nearly parallel spirals (*see* photograph, Fig. 3); on others there were two systems of parallel spirals of opposite hand. In a few cases there were, in the neighbourhood of the intersection of two bands, a number of other very short lines apparently running in random fashion at diverse angles; apparently these lines were produced subsequently to the main long lines, and were probably due to the irregular conditions caused by the disturbance due to the first two lines. The lines on any one specimen were not strictly parallel to each other; in Tables A, B, C and D the maximum and minimum slopes are given as well as the average.

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* It was not possible to observe the time of the appearance of the lines on the tubes tested under water pressure external to the tube, nor of the lines on the inside surface on any of the specimens.

The formation of the Lüders' surfaces does not seem to be a process of massive slipping. In some cases when the stress, at the first indication of yield, was increased very slowly and carefully, the lines could be seen slowly extending themselves in their spiral way round the tube. To watch the development of the lines or bands gave the impression of the propagation of a disturbance rather than of a process of massive slipping.

Measurement of the Angles of the Lines.

The angles of the lines were measured by wrapping paper on the tube. It was very difficult to measure the spiral angles of the inside lines; on account of this difficulty the angles of only a few are given in the Tables. Sometimes there were no lines at all on the inner surface; probably this may be accounted for by the difference in the kind of scale on the inside and outside. Again, occasionally no lines showed on the outside when the extensometer indicated the neighbourhood of the yield point, but were to be seen on the interior surface after the test.

Specimens on which the lines were not sufficiently distinct for the angle to be measured were rejected unless the stresses at yield were useful for comparison. A number of specimens showed no lines. A troublesome circumstance was that, although the lines on a few of the 11 G specimens were vividly distinct at the time of the test, yet after the lapse of a day they were scarcely visible.

In several cases, after a test with a given ratio of p_1 to p_2 had been made, and the lines had appeared, a test with another ratio of p_1 to p_2 was made, and a new yield point sought, on the same specimen. In the second test, the lines which had previously appeared broadened out and extended their length, but usually the new extension was at a different spiral angle. Sometimes entirely new lines would appear in the second test; these were always at a different inclination to those of the first test.

On account of the close and uniform agreement between the time of appearance of the lines and the commencement of the yield as shows by the extensometer, the author did not think it necessary to use the extensometer always, and accordingly many of the tests were made without the aid of that instrument.

One of the most interesting facts about the Lüders' lines was the occurrence of lines on the outside and the inside of a specimen, radially opposite, or very nearly opposite, to each other—*i.e.*, a radial line through the axis of the tube passed through both the inner and outer lines. This pairing of lines occurred on most of the specimens. The ends of a pair of pairs were also radially opposite.

That such "pairs" of lines or of bands were radially opposite was proved by cutting transversely a few of the tubes on which the lines were well marked on both surfaces, and scribing radial lines on the surface of the section. Another method was that of taking a point on an outside line and finding a point radially opposite to it on the inside surface by means of the scribing block and angle square. While it was not an easy matter to find the exact point where the centre line of a band or line intersected the edge of a cut section of tube, it was quite evident that the "circumferential displacement," if any, between an inner and outer line was quite small. Most of the lines appeared to be exactly opposite; there was apparently a small displacement in the case of a few lines, but only in three or four pairs of lines was there more displacement than could be accounted by error of measurement. It seemed that, most probably, local irregularity was the cause of the apparent displacement in these three or four cases. The operations for testing the radial positions of the lines were done with the greatest care, since upon them depended the determination of the planes or surfaces of yielding.

Determination of the Inclination of the Surfaces or Canals of Yielding from the Lüders' Lines.

Since pairs of lines, radially opposite on the tube surfaces, were always found (when, as in many specimens, there were lines on both surfaces) it is evident that a pair of lines or bands so situated must be the traces of the same spiral surface or canal. The absence of "circumferential displacement" between a pair of lines together with the fact that the ends of a pair were also opposite to each other indicate that a plastic movement, commencing probably at the inner surface, first makes its way through the tube wall, then extends in a spiral direction occupying in its advance the whole thickness of the tube wall. An inside line does not appear, therefore, to precede an outer line except at the commencement; and the process of spreading of the disturbance will not be, as would naturally be expected, a successive yielding from the inner surface outwards. If the process of yielding consisted in the penetration through the wall thickness of long lines first formed on the inner surface (where initially the stress is a maximum) the inclination of the lines on both surfaces would be the same.

Since this is not the case, the lines being radially opposite, the inclination of inside and outside surface lines are different; and measurements showed this difference to be that of two spiral lines, one on either surface, which make a complete revolution of the tube in the same axial length. Moreover, being radially opposite, a pair of lines must be the traces of a spiral surface generated by the movement of a radial line which always intersects a helix. In the appendix to this Paper the case of a plane skew to the wall of a 10 G tube is considered, the wall, however, being supposed plane instead of cylindrical. If the traces of such a plane be inclined at an angle a (not differing from 45 deg. by more than ±5 deg.), it is shown that there will be a "displacement" from the normally opposite position, which will be unmistakably measurable if the inclination of the skew plane itself differ as little as 0·75 deg. from *a*.*

Applying this to the cylindrical 10 G tube, it is evident that the Lüders' surfaces may be taken to be, without serious error, right spiral surfaces.

In view of the process of formation of the Lüders' surfaces, and also of the foregoing considerations, it appears to the author that, for the purpose of comparison of the inclinations of the surfaces in tubes of different thicknesses, the inclination at the mid-wall of the tube should be taken, rather than the angles of the traces. By doing this, the angles for the tubes of different thicknesses agree very fairly. Moreover, it seems reasonable to consider the mid-wall angle, rather than the angle on the inner surface as being the probable inclination of the Lüders' surfaces had not the circular form of the wall caused a variation from surface to surface. Accordingly, in the Tables A, B, C and D, the angles of the Lüders' lines, measured on the surfaces, have been reduced to the corresponding angle at the mid-wall by adding or subtracting to a surface angle half the difference of inclination between that angle and that of another spiral line radially opposite on the other surface.

On most of the tubes tested in simple compression there were lines or bands running circumferentially. These lines were not so long or so regular as the inclined lines, and the scale usually flaked somewhat on either side, making a ragged line. There were often lines, corresponding to them in position, or the internal surface; a pair of such lines were evidently the traces of a surface of yielding. There was always a longitudinal displacement between the inner and outer of a pair of these lines, the displacement being about equal to the wall thickness. But the lines were too ragged to obtain an accurate measurement of the displacement, and therefore the angle of the conical surface of yielding could not be obtained.

State of Stress during the Propagation of the Lüders' Lines.

Before any non-elastic strain has taken place the hoop stress will vary from a maximum at the inner surface of a tube to a minimum at the outer, this stress at radius r being given by the formulæ :—

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* *See* remark in the Appendix, p. 324, concerning the thinner tubes.

For internal pressure p_i,

$$p_r = -\frac{p^i \,.\, r_i^2\left(1+\frac{r_e^2}{r^2}\right)}{r_e^2-r_i^2},$$

For external pressure p_e

$$p_r = \frac{p_e r_e^2\left(1+\frac{r_i^2}{r^2}\right)}{r_e^2-r_i^2},$$

where p_r is the hoop stress at radius r,
r_i is the internal radius,
r_e is the external radius.

Thus, for internal pressure the hoop stress varies from $p_i\frac{r_i^2+r_e^2}{r_e^2-r_i^2}$ at the inner surface to $p_i\frac{2r_i^2}{r_e^2-r_i^2}$ at the outer surface ; a variation of about 10 per cent. in the thickest tubes and about 5 per cent. in the thinnest tubes.

The inner surface is also the locality of greatest shear stress, this being half the difference between the hoop stress and the longitudinal stress. The first breakdown of elasticity will no doubt occur at some point or points of the inner surface, but at a very much lower stress than that at the yield point. Mons. F. Osmond* has remarked " that the stress-value of the elastic limit depended in crystalline bodies upon the crystalline orientation in comparison with the direction of the stress " ; thus ferrite crystals disadvantageously situated with respect to the stress direction should suffer elastic breakdown first. Also, considered microscopically, the structure of mild steel is heterogeneous, the weaker ferrite crystals being mingled with the harder pearlite. A conclusion reached by Mr. L. Bairstow from his experiments on cyclical variations of stress is that there is elastic breakdown under a static stress very much less than the yield point stress, though the aggregate amount of the strain is too small to be detected by the extensometer.†

As the number and probably the extent of the localities of plastic strain‡ increase with increment of stress, this strain will begin to be registered by the extensometer; and the stress-strian curve will show signs of departure from the straight line. The point of departure—if point it can be called, since its position is indefinite—is what is usually regarded as the "elastic limit." Between this "elastic limit" and the yield point quite a large stress difference is sometimes found; but in tests wherein care is taken to obtain uniformity of applied stress this difference is small—as was the case in nearly all the author's tests. The opinion has been expressed by some writers* that could perfectly uniform stress be imposed on a homogeneous test specimen of mild steel, the elastic limit and the yield point would be found to coincide. In view of the nature of mild steel and wrought iron the author ventures to think that the proximity of the extensometer "elastic limit" and the yield point must be due to some other cause than uniform stress distribution and homogeneity of the material.

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* " Proc." Inst. Mech. Eng., Vol. IV., 1909, p. 1303.

† " Iron and Steel under Cyclical Variations of Stress."—National Phys. Lab. " Collected Researches," Vol. VI., 1910, p. 181.

‡ Dr. Rosenhain was unable to find by microscopic examination evidence of any other kind of non-elastic strain other than that of cleavage slipping.—" Journal," Iron and Steel Institute, Vol. LXX., 1906.

In the author's tests of tubes the sharp bend in the stress-strain diagram—*i.e.*, the yield point—coincided with the appearance and spreading of the Lüders' lines. A noticeable circumstance, also, was the small aggregate amount of non-elastic strain indicated by the extensometer (Figs. 2*a*, 2*b*) when the lines appeared, but the comparatively large augmentation as the lines extended. Evidently the process of yielding consisted in the propagation of plastic strain, as manifested by the Lüders' lines; such spreading being, doubtless, into regions previously affected by plastic strain but of so limited amount or partial distribution that the aggregate effect indicated by the extensometer was very small.†

As before described, the lines spread on both surfaces simultaneously, and the advancing front of a line of yielding extended across the width of the tube wall. The stress in the material immediately proximate to the advancing front must, for obvious reasons, be incalculable; but the result was to give Lüders' lines, which, on the whole, had regular and definite inclinations. The fact that the lines usually appeared at an end of a specimen did not vitiate this regularity, except, in some cases, close to that end. The applied loadings at the yield point were undoubtedly not those initiating elastic breakdown, but were those under which the plastic strain propagation became possible, or, at any rate, was actually effected.

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* Dr. J. Muir, "On the Overstraining of Iron by Tension and Compression." "Proc." Roy. Soc., A 77, 1896, p. 278.

Mr. J. J. Guest, "Ductile Materials under Combined Stress." "Proc." Phys. Soc., Vol. XVII., p. 208.

† Mr. G. T. Beilby's conclusions concerning the formation, between slipping cleavage surfaces, of mobile material which almost immediately hardens into a vitreous or amorphous phase has an important bearing on the whole question of the yield point, and very probably on the whole strain history from the crystalline elastic limit to the yield point.

See "Proc." Roy. Soc.; also "Engineering," May 19, 1911, p. 660.

The maximum stresses in the tubes (at the inner surface), calculated on the assumption that they were still perfectly elastic at the commencement of propagation of the large plastic yield strain, are the stresses given in the Tables A, B, C, D. They may be regarded as approximately the yield point stresses—*i.e.*, the stresses under which rapid extension of plastic yielding was precipitated.

The tabulated stresses show the same result of other experiments on combined stress—viz., that the maximum shear stress in the material is approximately a constant at the yield point. The main use of the calculated stresses has been to find from them the ratio : longitudinal stress/hoop stress, a ratio that was given a wide range in the experiments. Uncertainties in the stresses make a difference in the ratio that is small in comparison with the extent of its range.

The Applied Stresses and the Directions of the Lüders' Surfaces.

It will be seen from Tables A, B, C, D that there is, with a few exceptions, a regular variation of the inclination of the Lüders' lines corresponding to the ratio between the longitudinal stress and the hoop stress.

The total variation of angle in the author's tests was about 8 deg.—viz., from about 50 deg. (to the axis)—with end pressure only, to about 42 deg. with hoop tension four times the longitudinal compressive stress. With these stresses equal in intensity, the angle was 45 deg. or very nearly. The angles just quoted are the angles on the interior or exterior surfaces reduced to the corresponding angle at the mid-wall of the tube (*see* p. 318). Since it was possible that the alteration of angle might be due to the tubular form of the specimens and the particular circumstances of the testing, the last four tests of Table D were made. In the latter water pressure was applied to the exterior surface simultaneously with longitudinal tension—*i.e.*, the stresses were reversed from the directions in the other tests. The results show the same or a very similar relation between the angles of the lines and the directions of the stresses—*e.g.*, the lines were inclined at about 48 deg. to the direction of the hoop compression (unaccompanied by longitudinal tension), although they were now at 42 deg. to the axis of the tube. These results show that the Lüders' lines and surfaces do not follow lines of weakness due to, or are materially affected by, the process of manufacture; and they also indicate that the directions of the lines are not affected by instability ("secondary flexure") of the tube walls under the applied forces. Confirmation of the latter is the similarity of results from tubes having different ratios of thickness to diameter (*see* Tables). Also these results are some evidence that the form of the specimens allows (comparatively) free development of the lines and surfaces of yielding. Had the tube walls been thicker (or the specimens solid and strained in torsion) it is possible that the whole thickness of wall would not have been brought to the yielding condition at the same time, and that the propagation of the strain would not have been so free; for even with the thicknesses experimented upon the angle of a Lüders' surface varied through the thickness.

Conclusions.

The yielding of mild steel tubes under longitudinal and hoop stresses of opposite sign consists in rapid spreading of plastic strain along lines of definite inclination to the axis of the tube. The mean inclination of the lines or surfaces of interior disturbance is the same as the mean of the surface Lüders' lines (with the apparent exception of the lines at 90 deg. in tests in end pressure only.)

With longitudinal stress p_1 and hoop stress p_2 (of opposite sign to p_1) the angle of inclination α and the ratio p_1/p_2 show variations which are correlated thus:—

α is 45 deg.	when	$p_1/p_2=1$.
α increases from 45 deg. to about 50 deg.	while	p_1/p_2 varies from 1 to ∞.
α decreases from 45 deg. to about 42 deg.	while	p_1/p_2 varies from 1 to 0·25.

The inclinations are the same to the direction of a simple stress whether the stress is a push or a pull; and given two stresses of opposite sign at right angles to each other, and of unequal intensity, the lines were more inclined to the direction of the larger stress, whether this was tensile or compressive.

These corresponding variations in α and in p_1/p_2 seem to be related as cause and effect.

Assuming this to be the case, it would appear that a shear stress of a certain maximum intensity will not be the only criterion for the condition of yielding. If a (half) difference of the principal stresses of given amount be the only condition for propagation of the yield plastic strains, the author fails to see why there should be any alteration of inclination of the Lüders' surfaces so long as this given amount obtains, whatever the relative intensities of the principal stresses. Given the qualitative effect of change of inclination, the stress quantitative effect can hardly be negligible.

Application to other Tests of Mild Steel and of Wrought Iron.

Essential conditions for good development of the Lüders' lines appear to be that the dimensions of the specimen perpendicular to the direction of the greatest and least principal stresses should be small in comparison with the other two dimensions ; and that the surface of the specimen should be in a state such that the traces of the internal disturbances may become visible.

The latter condition is often fulfilled by the scale on the specimen, as in the author's tubes (though occasionally a batch of tubes would show the lines so feebly as to necessitate their rejection). Surface polishing of the specimens also causes the lines to show clearly.

In specimens on which the lines either do not appear or only appear feebly or irregularly, it seems highly probable that the yielding—well defined and rapid as it is when care is taken to have uniform distribution of stress—will nevertheless be a process similar in kind to that of the yielding of the author's tubes.

In this connection the remarkable conclusion of Mr. L. Bairstow has an interesting bearing—viz., that " below the static yield point, iron and steel appear to be capable of maintaining an untsable condition against cyclical variations of stress which ultimately produce a considerable change of length. The first application of the maximum stress may show only a scarcely measurable extension, in spite of the fact that extension of thousands of times the amount may be obtained without any change in the cycle of stresses."*

On the other hand, for static (*i.e.*, gradually applied) stress, it would appear that increasing stress brings the specimen into a state of instability, which culminates in the rapid breakdown of the yield. Under this view, yield point stresses are those not appertaining to any elastic limit, but are those which precipitate rapid yielding, from an unstable condition, of material which has already experienced a history of elastic breakdown.

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* " Iron and Steel under Cyclical Variations of Stress." National Phys. Lab., " Collected Researches," Vol. VI., 1910, p. 181.

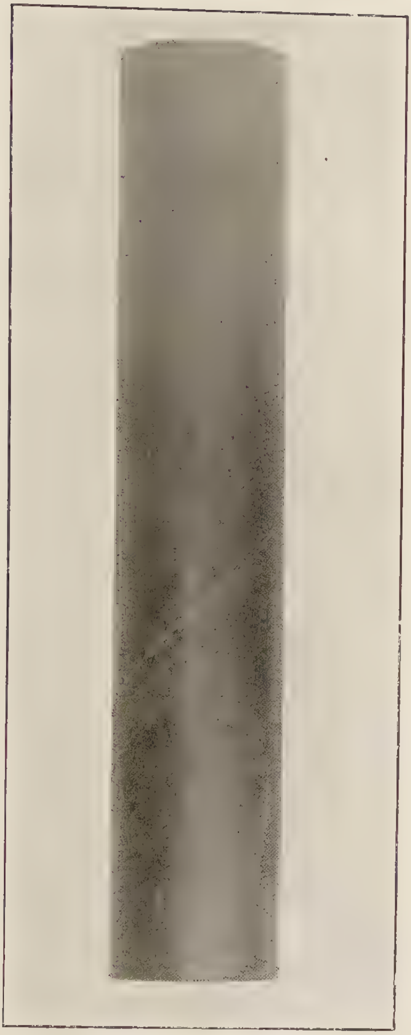
If it be assumed that the inclinations of the Lüders' surfaces of these experiments are those of unrestricted development (*see* p. 319) of the process of the yielding, specimens exhibiting irregular lines of indefinite inclination would apparently have yielded under conditions which hampered the process, *e.g.*, the neighbourhood of material, in a thick tube or solid specimen, not yet stressed to a condition of instability. In such a case the yield point would probably be retarded, and the yield stresses somewhat higher.

The expense of apparatus and material for the foregoing experiments was defrayed partly by a grant from the University of Liverpool and partly from a grant from the " Bryan Donkin " fund made by the Institution of Mechanical Engineers. The work was done in the Walker Engineering Laboratory of the University of Liverpool ; and the author desires to thank the Director of the Laboratory, Prof. W. H. Watkinson, for affording facilities therein for carrying out the tests. He also wishes to thank Mr. W. Donaldson, the Chief Instructor of the Laboratory, for many valuable suggestions, and he is grateful to those students who occasionally assisted him with the tests.

APPENDIX.

The Relation between the Inclinations of the Lüders' Lines and those of the Surfaces of Yielding.

The experiments described in the foregoing Paper show the Lüders' lines to be inclined to a generator at angles varying from about 50 deg. to about 42 deg., according to the stress system in force when the metal yielded. Assuming, for the argument of this appendix, that the lines are the traces



Specimen X. 12. Table C.
Inclination (to axis) of lines on exterior surface between $47\frac{1}{4}°-48\frac{3}{4}°$.

FIG. 3.

To face p. 324].

TABLE A.—*Cold Drawn Mild Steel Tubes. Annealed, except where otherwise stated.*

Bore, thickness, &c.	Specimen.			Yield point, principal stresses.‡			Maximum shear stress.	$\frac{P_1}{P_2}$	Inclination to axis of tube of the Lüders' lines.								Remarks.
				Tons per square inch.													
	No.	Test.*	Length.	Long. P_1	Hoop P_2	Radial P_3			Surface on which the lines are measured.	Range of variation in inclination.	Number of lines measured.	Average angle of inclination of lines on exterior surf.	Average angle of inclination of lines on interior surf.	Average angle reduced to corresponding angle at mid-wall of tube.§ $a_o-0·75°$ $=a_i+0·75°$			
				Compressive stress + *ve*.			Tons per sq. in.					a_o	a_i		$=a$		
3 in. bore, 14 Gauge, about 0·08 in. thick.	II.B1	...	8·75″	4·50	−13·50	0·645	9·00	0·33	Interior	42·0°–43·0°	Several	...	42·2°		42·95°	43·6°	......
	II.22	1	12·0″	3·60	−10·80	0·52	7·20	0·33	Exterior	44·5°–45·5°	2	45·0°	...		44·25°		
	II.B4	2	8·75″	4·25	−13·50	0·645	8·87	0·32	Lines of former test broadened and extended								
	†II.9	...	7·25″	3·80	−7·80	0·375	5·80	0·5	Interior	42·6°–44·3°	4	...	43·4°		44·1°		......
	†II.3	1	15·0″	6·78	−6·74	0·32	6·76	1·0	Interior	...	1	...	45·0°		45·75°	45·5°	......
	†II.5*b*	...	7·0″	7·08	−7·42	0·355	7·25	0·96	Interior	All about 44°	Several	...	44·0°		44·75°		
	II.21	2	15·0″	7·80	−7·80	0·37	7·80	1·0	Exterior	45·0°–47·0°	4	46·3°	...		45·55°		
	†I.2*a*	2	18·0″	8·20	−8·10	0·385	8·15	1·0	Interior	Both at 45°	2	...	45·0°		45·75°		
	†I.5	1	7·0″	8·30	−8·05	0·38	8·17	1·03	Exterior	46·0°–47·0°	3	46·5°	...	45·75°	45·25°		
									Interior	All about 44°	Several	...	44·0°	44·75°			
	I.11	2	9·0″	9·25	−9·25	0·47	9·25	1·0	Exterior	46·0°–46·4°	2	46·2°	...		45·45°		
	VII.3	...	4·0″	12·7	−13·8	0·65	13·25	0·92	Exterior	46·5°–47·5°	Several	47·0°	...		46·25°		Not annealed
	VII.4	...	7·0″	12·9	−14·10	0·66	13·50	0·92	Exterior	All about 46°	Several	46·0°	...		45·25°		Not annealed
	†V.2	...	6·1″	11·25	−11·25	0·625	11·25	1·0	Exterior	46·5°–47·5°	Several	47·0°	...		46·25°		Not annealed
	†V.3*a*	...	19·0″	11·10	−11·10	0·61	11·10	1·0	Exterior	All about 46°	Several	46·0°	...		45·25°		Not annealed
	†II.8	1	7·25″	8·80	−4·40	0·217	6·6	2·0	Interior	45·0°–46·5°	Several	...	46·0°		46·75°		......
	VII.5	...	7·0″	16·63	−9·24	0·435	12·93	1·8	Angles not measurable			...	...		......		Not annealed
	II.24	...	6·1″	10·73	−3·81	0·182	7·27	2·8	Exterior	50·0°–52·0°	3	51·0°	...	50·25°	49·0°	48·2°	......
									Interior	47·0°–51·0°	3	...	48·7°	49·45°			
	I.11	1	9·0″	13·42	−4·58	0·232	9·00	2·9	Exterior	Both at 48·25°	2	48·25°	...		47·5°		
	II.22	2	12·0″	10·9	−2·56	0·122	6·73	4·25	Exterior	47·4°–50·2°	3	48·9°	...		48·15°	49·0°	......
	II.B4	1	8·75″	14·4	−2·82	0·135	8·60	5·1	Exterior	48·7°–49·2°	3	49·0°	...		48·25°		
	II.21	1	15·0″	12·3	−2·56	0·122	7·43	4·80	Exterior	51·0°–52·0°	2	51·5°	...		50·75°		

* Where more than one test has been made on the same specimen, a number denoting the order of the making of the test is given in this column.
† The yield point stresses (but not the angles of the Lüders' lines) of specimens marked † were given in a Paper by the author, "Proc." Inst. Mech. Eng., 1909, No. 4.
‡ These are the stresses at the interior surface of the tube, calculated, on the assumption that the tube is perfectly elastic, from the loads at the yield point.
§ *See p.* 318.

TABLE B.—*Hot Drawn Mild Steel Tubes. Annealed.*

Bore, thickness, &c.	Specimen.			Yield point, principal stresses.‡			Maximum shear stress.	$\frac{P_1}{P_2}$	Inclination to axis of tube of the Lüders' lines.								Remarks.
				Tons per square inch.													
	No.	Test.*	Length.	Long. P_1	Hoop P_2	Radial P_3			Surface on which the lines are measured.	Range of variation in inclination.	Number of lines measured.	Average of angle of inclination for exterior surface.	Average of angle of inclination for interior surface.	Average angle reduced to corresponding angle at mid-wall of tube.§ $=a_o-1·25°$ $=a_i+1·25°$			
				Compressive stress + *ve*.								a_o	a_i		$=a$		
Specimens cut from same length of Tubing. (10 Gauge.) Bore 2¾ in. Thickness about 0·128 in.	†III.1	2	14·0″	8·43	−8·52	0·743	8·47	1·0	Exterior	45·3°–46·5°	5	45·9°	...	44·65°	44·95°	44·85°	......
									Interior	Several lines about		...	44·0°	45·25°			
	†III.2	2	9·0″	8·26	−8·30	0·725	8·28	1·0	Lines not measurable								
	†III.5	2	9·0″	8·12	−8·20	0·717	8·16	1·0	Exterior	...	1	45·7°	...		44·45°		
	†III.6	2	9·0″	8·28	−8·53	0·745	8·40	0·97	Exterior	45·5°–46·2°	2	45·9°	...		44·65°		
	†III.7	3	9·0″	8·55	−8·60	0·751	8·57	1·0	Exterior	...	1	46·8	...	45·55°	45·30°		
									Interior	...	1	...	43·8°	45·05°			
	†III.8	2	8·0″	8·05	−8·38	0·732	8·21	1·0	Lines of former test broadened								
	†VIII.2	2	4·5″	9·80	−9·70	0·847	9·75	1·0	Lines not measurable						......		Not annealed
	†III.3	1	4·5″	8·58	−8·60	0·751	8·59	1·0	Do. do.								
	III.10	2	4·1″	8·15	−8·15	0·713	8·15	1·0	Do. do.								
	†III.7	2	9·0″	10·41	−5·86	0·512	8·13	...	Do. do.								
	†III.8	1	8·0″	11·74	−4·81	0·420	8·27	2·44	Exterior	46·8°–48·6°	6	47·9°	...		46·65°	47·3°	......
	†III.9	...	8·0″	11·50	−3·82	0·335	7·66	3·0	Exterior	46·8°–49·4°	6	48·5°	...		47·25°		
									Interior	...	...		About 46·0°				
	III.11	...	4·8″	11·86	−3·60	0·315	7·73	3·3	Exterior	48·4°–50·0°	4	49·2°	...	47·95°	47·85°		
									Interior	46·5°	2	...	46·5°	47·75°			
	III.10	...	4·1″	12·80	−3.20	0·280	8·00	4·0	Exterior	...	1	49·0°	...	47·75°	47·60°		
									Interior	46·2°	2	...	46·2°	47·45°			
	†III.1	1	14·0″	16·1	0	0	8·05	∞	Interior	About 50°	...	...	...		......		Also lines at 90°
	†III.2	1	9·0″	16·2	0	0	8·10	∞	Exterior	47·3°–52·0°	3	50·5°	...		49·25°	49·65°	Do. do.
	†III.5	1	9·0″	14·78	0	0	7·39	∞	Exterior	49·0°–52·0°	3	50·5°	...		49·25°		Do. do.
	†III.6	1	9·0″	15·83	0	0	7·91	∞	Exterior	...	1	52·5°	...		51·25°		Do. do.
	†III.7	1	9·0″	16·10	0	0	8·05		Exterior	48·0°–53·0°	5	51·0°	...		49·75°		Do. do.
									Interior	About	48·5°						
	†VIII.2	1	4·5″	16·10	0	0	8·05	∞	Exterior	49·5°–50·5°	2	50·0°	...		48·75°		Do. do. Not annealed

* Where more than one test has been made on the same specimen, a number denoting the order of making the test is given in this column.
† The yield point stresses (but not the angles of the Lüders' lines) of the specimens marked thus † were given in a Paper by the author, "Proc." Inst. Mech. Eng., 1909, No. 4.
‡ These are the stresses at the internal surface of the tube, calculated, on the assumption that the tube is perfectly elastic, from the loads at the yield point.
§ *See* p. 318.

To face p. 324.

TABLE C.—*Cold-Drawn Mild Steel Tubes. Annealed.*

Bore, wall-thickness &c.	Specimen.			Yield point, principal stresses.* Tons per square inch.			Maximum shear stress.	$\frac{P_1}{P_2}$	Inclination to axis of tube of Lüders' lines on exterior surface.				Remarks.
	Number.	Test.	Length.	Long. P_1	Hoop P_2	Radial P_3			Number of lines measured.	Range of variation in inclination on specimen.	Average of angles of inclination. α_o	Average angle reduced to corresponding angle at mid-wall of tube $=\alpha_o-1\cdot25°$.†	
				Compressive stress + *ve.*									
Specimens cut from same length of tubing 2·5 in. bore. 11 Gauge (about 0·118 in. thick).	X.1	...	8·0″	3·4	−10·2	0·88	6·80	0·33	3	45·2°–46·6°	45·7°	44·45°	......
	X.7	...	8·1″	5·2	−10·4	0·87	7·80	0·5	3	45·5°–47·0°	46·3°	45·05°	......
	X.6	...	8·1″	6·6	−6·8	0·57	6·70	1·0	2	46·5°–48·0°	47·3°	46·05°	......
	X.10	1	8·1″	8·56	−8·8	0·735	8·68	0·97	2	46·0°–46·2°	46·15°	44·9°	......
	X.11	2	7·9″	7·96	−8·0	0·665	7·98	1·0	2	46·0°–47·2°	46·6°	45·35° } 45·5°	......
	X.13	2	9·0″	8·05	−8·0	0·665	8·02	1·0	3	47·0°–47·7°	47·4°	46·15°	......
	X.14	2	5·8″	8·15	−8·20	0·67	8·17	1·0	1	...	46·0°	44·75°	......
	X.2	...	8·1″	9·4	−4·7	0·395	7·05	2·0	2	47·2°–48·8°	48·0°	46·75° } 47·1°	......
	X.9	...	8·0″	9·8	−4·9	0·42	7·35	2·0	1	...	48·7°	47·45°	......
	X.3	...	8·1″	12·6	−4·2	0·35	8·4	3·0	2	47·7°–48·2°	48·0°	46·75°	
	X.5	...	8·0″	9·3	−3·1	0·26	6·2	3·0	4	49·5°–51·0°	50·3°	49·05° } 48·1°	......
	X.8	...	8·1″	9·6	−3·2	0·26	6·4	3·0	2	50·3°–51·6°	51·0°	49·75°	
	X.12‡	...	15·0″	12·0	−4·0	0·335	8·0	3·0	3	47·2°–48·8°	48·1°	46·85°	
	X.10	2	8·0″	12·4	−3·2	0·26	7·8	4·0	Lines of first test broadened				
	X.11	1	7·9″	12·4	−3·1	0·26	7·75	4·0	2	48·0°–49·2°	48·6°	47·25°	......
	X.13	1	9·0″	12·73	−2·30	0·191	7·52	5·5	3	49·5°–50·0°	49·7°	48·45° } 48·0°	......
	X.14	1	5·8″	13·25	−2·32	0·193	7·8	5·7	2	48·7°	48·7°	47·45°	

* These are the stresses at the internal surface of the tube calculated on the assumption that the tube is perfectly elastic, from the loads at the yield point.

† *See* p. 318. ‡ Illustrated in Plate Fig. 3.

TABLE D.—*Specimens Bored and Turned from Hot-Drawn Mild Steel Tubing.*

For detail of heat treatment after lathe-work *see* p. 308.

	Specimen.			Load on testing machine.	Water pressure.	Longitudinal load due to water pressure.	Net longitudinal load on specimen.	Yield point, principal stresses, tons per sq. in.‡			Maximum shear stress.	$\frac{P_1}{P_2}$	Inclination to axis of tube of Lüders' lines on exterior surface.				Remarks.
	No.	Length.	Mean thickness of tube wall.					Long. P_1	Hoop P_2	Radial P_3			Number of lines measured.	Range of inclination on specimen.	Average angle of inclination. α_o	Average angle reduced to corresponding angle at mid-wall of tube. $=\alpha_o-1\cdot0°$*	
								Compressive stress + *ve.*									
				Tons.	Lb. per sq. in. *Internal*	Tons.	Tons.				Tons per sq. in.					$=\alpha$	
Cut from same length of Tubing. Bore 2¾ in. Thickness ¼ in.	2	8·0″	0·099″	11·19	2,530	−7·29	3·90	4·23	−17·05	1·13	10·64	0·25	2	43·6°	43·6°	42·6° } 41·8°	
	9	4·0″	0·0985″	12·56	2,870	−8·25	4·31	4·70	−19·62	1·28	12·16	0·24	4	41·5°–43·5°	42·0°	41·0°	......
	3	8·0″	0·098″	15·72	1,740	−5·00	10·72	11·75	−11·75	0·78	11·75	1·0	6	43·0°–46·8°	45·0°	44·0°	......
	1	8·0″	0·098″	17·51	435	−1·25	16·26	17·85	−2·97	0·195	10·41	6·0	5	47·2°–47·6°	47·4°	46·4°	......
	0	4·0″	0·0985″	14·10	270	−0·78	13·32	14·55	−1·96	0·121	8·26	7·4	1	...	46·5°	45·5° } 45·3°	
	8	4·0″	0·0985″	15·70	310	−0·84	14·81	16·15	−2·09	0·138	9·12	7·7	2	45·6°–46·6°	46·1°	45·1°	......
	4*a*	3·15″	0·0985″	15·26	0	0	15·26	16·60	0	0	8·30	∞	1	...	48·6°	47·6°	......
					External												
	4	4·75″	0·099″	...	2,820	...	0	0	19·00	0	9·50	0	2	43·0°–43·5°	43·3°	42·3°	......
	6	9·75″	0·089″	...	3,050	(−4·00)¹	(−4·00)¹	(−4·80)¹	(22·9)²	0	(13·85)	(0·21)	7	41·3°–45·0°	43·3°	42·3° } 41·7°	......
	7	9·8″	0·0885″	...	3,020	(−3·95)¹	(−3·95)¹	(−4·80)¹	(23·4)²	0	(14·1)	(0·205)	18	39·3°–45·0°	42·2°	41·2°	Collapsed at 3,240 lb. per sq. in. external press.
	5	9·75″	0·0895″	−13·6	1,220	(−1·59)¹	(−15·19)¹	(−18·3)¹	(9·12)²	0	(13·7)	(2·0)	2	46·7°–47·0°	46·8°	45·8°	Central part of tube in the jacket.
				−13·6	0	0	−13·6	−16·4	?³	0	...	...	2	50·5°–52·5°	51·5°	50·5°	Ends projecting from the jacket.

1. Loads and stresses doubtful on account of the uncertainty of action of the cup leathers in the jacket apparatus, Fig. 1.
2. Stresses doubtful on account of the influence of the ends of the specimens, which projected beyond the jacket.
3. Hoop stress of unknown (probably small) amount due to the influence of the water pressure over the central 4½ in. of the specimen.

‡ These are the stresses at the interior surface, calculated from the loads at the yield point on the assumption that the tube is perfectly elastic.

* *See* p. 318.

on the tube wall of surfaces of yielding, these surfaces are not those on which the shear stress is greatest, except (presumably) in the case where the lines are at 45 deg. to a generator. The surfaces of maximum shear stress in the material, of which the last-mentioned lines may be the traces, are, of course, planes containing a normal to the tube wall and being also at 45 deg. to a generator.

An explanation of the variation of angle (which presented itself during the course of the experiments) was that the material might perhaps be giving way on planes or surfaces of weakness in the tube in a manner like the following. A Lüders' lines not at 45 deg. might be the trace of a plane B (whose inclination is nevertheless at 45 deg., or thereabouts, to a generator) which does not contain a normal to the cylindrical surface. The shear stress on B, though less than that on a plane A of maximum shear stress, might be sufficient to overcome the shearing resistance on B; and the stress on A, though greater than on B, might not be great enough to overcome the greater resistance to shear along A. Let Fig. 1 represent a plane section of the tube wall (supposed straight instead of curved for the sake of clearness), the section being normal to a generator. All such planes as B will cut this section in some line *cd*; let φ be the angle between *cd* and a normal *ab* to the tube wall.

Let each plane of the series B be fixed by its trace *cd* and its inclination to a generator. Let the principal stresses be p_1, p_2, p_3, where p_1 is parallel to a generator (the longitudinal stress), p_2 is perpendicular to a generator, and tangential to the tube wall (the hoop stress), p_3 is normal to the tube wall.

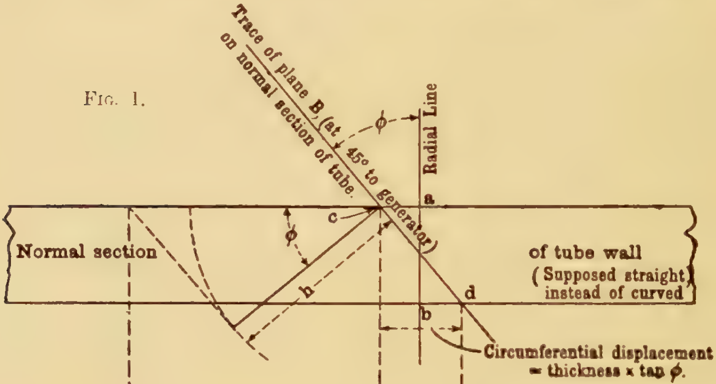
Then of all the planes whose traces *cd* make the same angle φ with *ab*, the one on which the shear stress is greatest is inclined to p_1 at an angle μ such that

$$\cos^2\mu=\frac{1}{2}\frac{(p_2-p_1)^2\cos^2\varphi+(p_3-p_1)^2\sin^2\varphi}{\{(p_2-p_1)\cos^2\varphi+(p_3-p_1)\sin^2\varphi\}^2}.$$

The inclination μ of the plane of maximum shear for any value φ may thus be found for any system of the principal stresses, p_1, p_2, p_3. Thence the intensity of this maximum shear stress on the plane may be found in the usual way.

The traces of these planes on the two surfaces of a tube will be (as shown in Fig. 2) at a distance from each other, measured circumferentially along the mid-wall surface, nearly equal to thickness × tan φ. This distance has been called the "circumferential displacement" in the foregoing Paper. The above formula facilitates the finding of the plane of maximum shear stress given the displacement between a pair of its traces on the inner and outer surfaces of a tube of given thickness subjected to given values of p_1, p_2, and p_3. The angles μ of such planes, and the comparative values of the shear stress for certain values of φ are given in Table E for each of certain stress systems. Also the angles θ which the traces of these planes on the mid-tube wall make with a generator are given in the Table, θ being calculated from the relation

FIG. 1.


Trace of plane B, (at 45° to generator)
on normal section of tube.
Radial Line
φ
a
c
φ
h
b
d
Normal section
of tube wall
(Supposed straight instead of curved)
Circumferential displacement = thickness × tan φ.

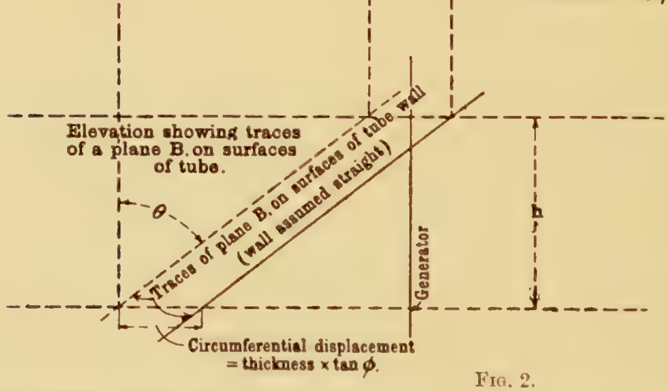

Elevation showing traces of a plane B. on surfaces of tube.
Traces of plane B. on surfaces of tube wall
(wall assumed straight)
θ
h
Generator
Circumferential displacement = thickness × tan φ.

FIG. 2.

$$\tan\theta = \frac{\tan\mu}{\cos} \text{ (see Figs. 1 and 2 above).}$$

Table E illustrates that the maximum shear on planes, which, while at or about 45 deg. to a generator, are skew to

TABLE E.

Stress system.	Plane of maximum shear stress passing through the line $\phi=0°$.			Shear stress on the plane $\phi=0$, $\mu=\theta$.	Plane of maximum shear stress passing through the line $\phi=15°$.			Shear stress on the plane $\phi=0$, $\mu=\theta$.	Plane of maximum shear stress passing through the line $\phi=30°$.			Shear stress on the plane $\phi=0$, $\mu=\theta$.
	Inclination to generator $=\mu$.	Inclination of trace to generator $=\theta$.	Shear stress on plane.		Inclination to generator $=\mu$.	Inclination of trace to generator $=\theta$.	Shear stress on plane.		Inclination to generator $=\mu$.	Inclination of trace to generator $=\theta$.	Shear stress on plane.	
$p_1=-p_2=11{\cdot}5p_3$	45°	45°	100	100	44° 24′	45° 24′	98·3	99·99	42° 51′	46° 58′	92·8	99·76
$p_1=-4p_2=40p_3$	45°	45°	100	100	44° 55′	45° 55′	98·55	99·95	44° 42½′	48° 49′	96·4	99·11
$p_1=-8p_2=80p_3$	45°	45°	100	100	44° 58′	45° 57′	99·15	99·94	44° 55′	49° 1½′	97·45	99·01
$p_2=0,\ p_3=0$	45°	45°	100	100	45° 0′	46° 0′	100	99·94	45° 0′	49° 6′	100	98·97

the surface of the tube, differs very little from the maximum shear stress in the material. The traces of these planes, for the stress systems of the Table (E), have inclinations very similar to those of the observed Lüders' lines for the same stress systems in the experiments. If, however, those planes were the actal planes of sliding, it would be very curious that the tube structure should be such as to give the regular and progressive values of α found.* (*See* Tables A, B, C, D in the Paper.) But if the Lüders' lines were the traces of such planes, there would be the circumferential displacement thickness × tan φ between a "pair" (*see* page 316 of the Paper) of the traces. This displacement would be greatest in the thicker tubes when stressed, so that one principal stress is much greater than the other. In the 10 G tube specimen, No. III. 7 of Table B, α is (about) 50 deg. for several lines; if the slipping occurred on that plane of maximum shear (at 45 deg. to a generator) which gives 50 deg. as the slope of the trace on the mid-wall surface, φ would be 33 deg., and the displacement would be $0{\cdot}128 \times \tan 33° = 0{\cdot}08$ in. If we suppose 40 deg. to be the slope of the planes of slipping (or about the angle commonly supposed to be the angle of slipping for compression alone) then, to give a trace at 50 deg., φ would be 45° 50′, and the displacement 0·13 in. Now, displacements of such magnitudes are much too large to be mistaken; indeed, there appeared to be no displacement except in one pair of lines on the above tube, and that was about 0·04 in. But it was very difficult to measure displacement (if any) with accuracy, the limit of accuracy being about 0·03 in. for this specimen. If we take a displacement of 0·03 in., φ will be

$$\tan^{-1}\frac{0{\cdot}03}{0{\cdot}128} = 13°\ 10',$$

and to give $\alpha = 50°$, the inclination of the plane of slipping will be 49° 15′. It is clear, then, that the slope of the planes of slipping and the inclination of their traces on the mid-wall surface must be practically the same for the lines at 50 deg.† on this specimen. If m is the inclination of a plane of slipping whose mid-wall trace is inclined α, then

$$\tan(\alpha - m) = \frac{\tan\alpha(1 - \cos\varphi)}{1 + \cos\varphi \tan^2\alpha}.$$

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* α is the angle of the Lüders' lines to the tube axis reduced to the corresponding angle at the mid-wall surface (*see* page 318 of the Paper).

† There were also other lines, due to other stress systems, on this tube.

Taking φ, as above, equal to 13° 10′, $(a-m)$ has the approximate value 0·75° when a has any value between 40 deg. and 50 deg.; hence, in the absence of displacement more than 0·03 in., all the planes of slipping in the 10 G tubes must be at practically the inclination of the corresponding Lüders' lines.

The same conclusion was reached from examination of all the eight tubes of different thicknesses on which displacement was looked for. Taking 0·03 in. as the error of measurement of displacement in the thinnest tubes tested (14 G), then $\varphi=20{\cdot}5°$, and $a-m=1{\cdot}8°$ (about) when a does not differ more than ± 5 deg. from 45 deg. Thus, even in these thinnest tubes, the observed variation of a could not be explained by skewness of the planes of slipping.

ABSTRACT.

Previous investigations have shown that Lüders' lines on specimens of mild steel and wrought iron, strained in tension, are inclined at about 50 deg. to the axis of pull. For tests in compression the information available is not precise, and though the angle of the lines with the direction of the compression is commonly understood to be about 40 deg., some doubt has been thrown on this point.

The author had found previously that the lines are well developed on the surface of mild steel tubes. Since it was easy to obtain a compressive stress of practically uniform distribution in tubes under end pressure, while at the same time a hoop tensile stress could be induced by internal fluid pressure, the author confined his attention to the lines on tubular specimens. These were of mild steel, either hot or cold drawn, and most of them were] annealed. They varied in bore from 2½ in. to 3 in., and in thickness of wall from about 0·08 in. to 0·125 in. Four sets of tubes were tested under end loading, simultaneously (for the most part) with internal water pressure, these loadings being so arranged as to give at the yield point ratios of the longitudinal compressive stress to hoop tensile stress ranging from 0·25 to ∞.

The Lüders' lines on the outer surface appeared at the yield point indicated by the extensometer—*i.e.*, their appearance coincided with the commencement of the large "yield" strain. In all cases where there were lines on the inner and outer surfaces of a tube, an inner and outer line, and also the ends of these lines, were found to be radially opposite; showing that the lines were traces of surfaces or canals of disturbance which passed through the tube wall, and indicating, moreover, that the disturbance spread spirally onwards, and not outwardly from a line initially formed on the more severely stressed inner surface.

The inclination a of the lines to the axis of the tube was found to vary in the following way for tubes under end load and internal pressure:—

For $\dfrac{\text{longitudinal stress}}{\text{hoop stress}}=0{\cdot}25$, $a=42$ deg. (about).

,, ,, ,, $=1$, $a=45$ deg. (about).

,, longitudinal stress only, $a=50$ deg. (about).

For values between 0·25 and 1, and between 1 and ∞, α became progressively larger.

Some tests of tubes under external water pressure and longitudinal tension gave results similar to those above mentioned. The directions of the lines with respect to the axes of like stress were practically the same, though the inclinations to the axes of the tubes were now complementary to the values of α quoted above.

The conclusion is drawn that the Lüders' surfaces have the same or approximately the same inclination to an axis of simple pull or simple push. With stresses of opposite sign at right angles to each other the lines and surfaces are more inclined to the stress of greater intensity, and with equal intensities the surfaces are at about 45 deg. The author suggests that if a shear stress of given intensity be the only condition for that spreading of plastic strain which commences at the yield point, then there is no reason why there should be any variation in the angle of the Lüders' lines and surfaces. The variation suggests that not only the maximum shear stress—*i.e.*, the half difference of the greatest and least principal stresses—but also the magnitudes of the principal stresses are factors in producing the yielding condition in mild steel.

DISCUSSION.

Dr. **Chree** said Lüders' lines seemed to possess a double interest. If their first existence coincided in time with the ending of linearity in the stress-strain relation, their presence intimated that the ordinary mathematical equations had ceased to apply. In this connection their utility would be greater if the stress function supposed to determine the elastic limit were greatest at a point or small area visible to the eye, instead of, as in the present case, over the whole inner surface of a cylinder. The phenomenon seemed, however, also interesting in itself. The nature of the change in the material and the mode of propagation of the altered volume would be more easily studied if the altered material could be recognised in sections of the cylinders.

Prof. E. G. **Coker** asked whether the lines were a surface phenomenon produced by the stresses in the oxide.

Prof. C. A. M. **Smith** said that he differed from the opinion of Mr. Mason as to what actually took place at elastic breakdown. He thought there could be no doubt that the record of the appearance of the Lüders' lines on the tubes and the measurement of the angles of the spirals added considerably to our knowledge of what happened at the yield point. He thought the great point of the work was the relation of the spiral on the inner surface of the tube with that on the outer surface. While he thought that for the phenomena of elastic breakdown tubes were unsuitable, yet the specimens used for this investigation on Lüders' lines possessed the great advantage that the Lüders' lines could be traced both inside and outside the tube.

Mr. E. H. **Rayner** pointed out that the elastic constants in the two directions would probably be different, and that the compression might put on a screw shear resulting in lines only appearing in one direction.

Dr. A. **Russell** asked the author if his results threw any light upon the phenomena obtained when wires are drawn.

Mr. **Maclachlan** asked the author if he had ever tried polishing the surface of his tubes. His own experiments made on the compression of solids showed that the lines were not always spirals, but approximated to a section of the cylinder made by a plane inclined at 45 deg.

The following communications respecting this Paper have been received :—

Mr. G. H. **Gulliver** writes : Mr. Mason gives results which are in general agreement with those of Hartmann and of my own. The most important new observation is that with two unequal principal stresses, of opposite sign, the lines are more inclined to the direction of the larger stress, whether this is tensile or compressive, and that when the stresses are equal and unlike the inclination is 45 deg. to each. The latter observation is not in agreement with results obtained by the writer with twisted cylindrical bars, upon which the lines were found to be inclined at 50 deg. to the direction of principal tension, and the former differs from results obtained with bent primsatic bars; but these apparent contradictions only indicate the need for further experiments. Another interesting observation of Mr. Mason is that on reloading a tube which has been strained already beyond the yield point, the new lines always differ in inclination from the old ones. Is the inclination increased or decreased, and is the change always in the same direction? The general form of the surface of sliding—which in detail has irregularities on a scale corresponding with that of the structure of the metal—is easily determined in the more simple cases; but if the metal be supposed structureless, a helical line should not be the trace of a surface, but the extreme member of a family of helices of constant inclination. In the case of a thin tube, as Mr. Mason shows, the family is replaced by a helical surface of constant pitch, but a thick-walled tube might be expected to give a different result. With regard to the stated non-appearance of Lüders' lines on ordinary test bars, this is much more a question of the condition of the surfare of the metal than of the relative dimensions of the bar. The lines will not be seen on a rough surface or on an ordinary turned surface, but they are quite noticeable on most mild steel bars and plates if the scale has not been removed before testing. The most accurate observations are made upon highly polished surfaces, but satisfactory photographic reproductions of such surfaces are difficult to obtain. Mr. Mason does not mention the occurrence of the continuous propagation of deformation in his specimens, as contrasted with the discontinuous propagation visible as Lüders' lines. It has been stated by Fremont that when care is taken to ensure uniform loading the propagation is continuous, and that, conversely, a discontinuous propagation indicates irregularity of loading. The writer disagrees with this, and would be glad to know of any observation of Mr. Mason bearing on the point. Referring to the case of lead, it may be noted that Humfrey observed markings, on specimens *etched after straining*, which, as shown in his photographs, have an inclination of approximately 50 deg. to the apparent direction of tension.

Mr. F. ROGERS writes: Having made an exhaustive investigation of Lüders' lines from certain aspects, the following conclusions may help to throw light on the nature of these strain effects in steel and soft iron. The following is a quotation from my Paper "Microscopic Study of Strain in Metals"* (p. 16): "Several years ago Prof. Ewing and the author were engaged in studying the nature of Lüders' lines, which, as is well known, appear obliquely to the direction of stress upon a suitably prepared surface of certain metals when stressed beyond the elastic limit. With further loading a line becomes a band of increasing width, within whose limits the surface of the specimen is visibly ruffled. With moderate magnification the wave-length of these undulations in soft irons—Swedish and Low Moor—was found to be from three to five times the average breadth of crystal grains in the specimen. The author has since found a similar relation between the wave-length—"of ruffling"—and size of grains in both normal and overheated samples of steels of the series referred to above, upon loading them statically in tension. Further, it is found that the ruffling of the surface, observed in fatigued samples of the normal class, has also a similar relation to the structure. The crests follow very irregular lines on the observed surfaces; this, and the variability of the wave-length, point to the dependence of the form, dimensions, and indeed existence, of the undulations upon the microscopical heterogeneity of the material." These investigations were made upon tensile test strips, say 4 in. tested length and $\frac{1}{2}$ in. by $\frac{1}{8}$ in. section, with the whole of one face, 4 in. by $\frac{1}{2}$ in., perfectly prepared beforehand for microscopic examination, and lightly etched. At the moment when the extensometer (Ewing's pattern) indicated in the least degree that the elastic limit had been exceeded the load was removed, and a Lüders' line was always found. Within this " line"—really a band, of course—slip bands could always, though with great labour, be found, and so also could the " ruffling " mentioned above. Incidentally, it is interesting that the Lüders' lines on surfaces thus prepared are perfectly permanent—that is to say, they will not alter or disappear so long as the specimen is prevented from rusting. Some of my samples look almost perfect now although they are about nine years old. There seems to be no important divergence of opinion between Mr. Mason and myself in those cases in which we have approached the problem apparently from the same points of view.

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* "Journal" Royal Microscopical Society, 1907, pp. 14-18.

The AUTHOR, in reply to Dr. Chree, said the experiments do not, in his opinion, show that the state of the material at the appearance of the Lüders' lines is one of perfect elasticity. Only the appearance of lines on the outside surface of the tubes was observed; possibly if the starting of scale disturbance on the inner surface had been observed, the coincidence between the bend on the stress strain curve and such initial disturbance might have been still more striking. But it is very difficult to believe, considering the constitution of the material, that the yield point and the Lüders' lines are not the symptoms of a collapse from a state in which some grains would have had large cleavage slips but for the relief of stress afforded by the surrounding material. If this view is correct, the equations of stress depending on the assumption of perfect elasticity may hold with approximate accuracy for small volumes, though they would not be correct for a " point in the material." With regard to the suggestion that it would be interesting if the Lüders' disturbance could be made visible in and through the wall thickness, it would seem to be hopeless that any indication would be visible under the microscope on a cut section of a tested specimen; while an end surface is liable to the irregularities of stress applied to it. The best evidence of the through-wall character of the disturbance is the presence, on some of the Table D specimens, not merely of pairs of opposite bands, but also of opposite pairs of fine lines which occur in the centres of the bands. The lines were the nuclei from which the bands spread out laterally.

Replying to Prof. E. G. Coker, the author believed the coating of scale to give evidence of plastic strain in the metal, because of the correspondence of the indications of the extensometer and the scale. If the scale indicated merely its own elastic or rupture limit, and not the disturbance in the steel, it is difficult to account for the pairs of opposite lines.

In reply to Prof. C. A. M. Smith, the author could only endorse the remark concerning the practical difficulty of deciding upon the relation between the " elastic limit " and the yield point.

In reply to Mr. E. H. Rayner, the Author relies upon the latter half of the experiments of Table D, in which the axes of tension and compression were interchanged in position in the tubes, to show that differences in the material, such as those mentioned by the speaker, had no sensible effect on the directions of the lines. With regard to the " screw " effect, it is possible that some instability in the testing machine might give rise to a very small amount of torque. But the effect of torque would be to turn the axes of stress round about a radius of the cylinder, and therefore the inclination of one set of spiral lines would be increased, while those of opposite band on the same specimen would be diminished. Both sets of spirals on a specimen varied in the same way with change of stress system.

In reply to Mr. Maclachlan, the author did not polish any of his specimens. For the reasons given, he regarded the scale as a very delicate indicator. He intended, however, to test a few polished tube specimens.

With regard to the elliptical, and not spiral, lines on solid specimens, the Author would be extremely glad to have further details of the specimens and the testing. An elliptical line on a solid specimen meant a large area for a continuous surface of yielding.

In reply to Mr. G. H. Gulliver's remarks, the author writes that the angle of the lines on twisted cylindrical bars and on bent prismatic bars do not correspond with those on the tubes. The author agrees that further experiment is needed to explain this difference, as well, also, to elucidate the relation between the dimensions and form of the specimen and the angles of the surfaces of yielding. The Author does not intend it to be understood that, on re-loading a tube which had been strained to yield, the newly-formed lines had a different inclination when the stress system was similar to that of the first loading.* Only when the value of $\frac{p_1}{p_2}$ was altered from that in the first test was any change of angle observed. The propagation of strain in lines or bands does not appear to indicate irregularity in distribution of loading. Long lines—sometimes making a whole spiral revolution, or even more, of a tube—had inclinations sensibly constant over their lengths. Considerable variation of inclination would be expected if such irregular distribution were the cause of the strain propagation described in the Paper.

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* *See* p. 316.

XXXI. *A New Method of Approximate Harmonic Analysis by Selected Ordinates.* By SILVANUS P. THOMPSON, *F.R.S.*

RECEIVED MAY 11, 1911. READ JUNE 9, 1911.

1. The approximate harmonic analysis of periodic curves may be attained by a simple and accurate process, considerably shorter than any hitherto described. According to Fourier's theorem any periodic single-valued function of x may be expressed in terms of the series—

$$y=A_1 \sin x+A_2 \sin 2x+\ldots.+B_1 \cos x+B_2 \cos 2x+\ldots.$$

The process of harmonic analysis consists in finding the coefficients A_1, A_2,....B_1, B_2.... of the several terms of this harmonic series.

In the ordinary processes, derived from Fourier's integrals, a number of equidistant ordinates are measured on the curve. Each ordinate is then multiplied by the sine of the angle at which it stands (or by the sine of that angle multiplied by the numeric of the harmonic to be found), the average of these products over a whole period being half the amplitude of the harmonic sought. This process yields the sine component of the harmonic in question. By substitution of cosine for sine in the multiplication the amplitude of the cosine constituent may be found. A method systematised by grouping prior to multiplication by the sines (or cosines) was propounded in 1903 by Runge*; and a shortened process based on this plan was brought by the Author before the Physical Society† in 1904.

In the new method now to be explained no multiplication by sines or cosines is required. The sole process is the simple arithmetical averaging of selected ordinates, followed by certain further additions or subtractions.

The basis of the method lies in the easily-verified fact that if a series of $2n$ ordinates is measured at intervals apart of π/n, where n is the numeric representing the order of the harmonic, and if their values, *taken alternately positively and negatively*, are averaged over a whole period, the mean so obtained is either simply the amplitudes of that harmonic or else is the sum of the amplitudes of that harmonic and of those of certain higher harmonics—namely, those the ordinal numeric of which is an odd multiple of n—the elimination of which is explained lower down. For cosine components the series of $2n$ ordinates must begin (or end) at the beginning (or end) of the period. For sine components the series must begin at $\pi/2n$ from the beginning of the period. Thus, to ascertain the value of the amplitude of the fifth cosine harmonic, all that is necessary is to average the 10 ordinates erected at 0°, 36°, 72°, &c., taken alternately positively and negatively; or in symbols,

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* "Zeitschrift für Mathematik and Physik," XLVIII., p. 443.

† "Proc." Physical Society, XIX., p. 443; read December 9, 1904.

$$B_5=\tfrac{1}{10}(y_0-y_{36}+y_{72}-y_{108}+y_{144}-y_{180}+y_{216}-y_{252}+y_{288}-y_{324})$$

and, similarly, for the fifth sine component,

$$A_5=\tfrac{1}{10}(y_{18}-y_{54}+y_{90}-y_{126}+y_{162}-y_{198}+y_{234}-y_{270}+y_{306}-y_{342}).$$

This is on the assumption that there are no components of the

FIG. 1.

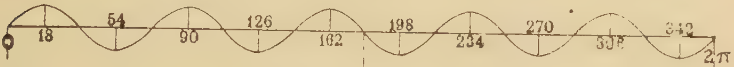


FIG. 2.

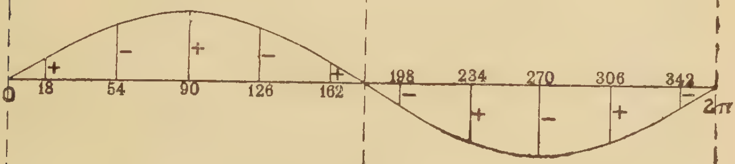
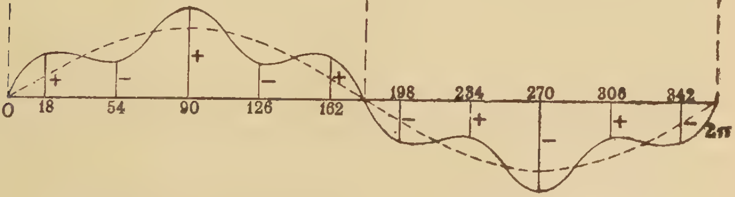


FIG. 3.



15th, 25th, 35th, &c., orders present. If such are present they must first be evaluated; for the true B_5 is the mean found above, minus $(B_{15}+B_{25}+B_{35}+$ &c.$)$. The similar correction for A_5 is to deduct from the mean, as found above, the sum of $(-A_{15}+A_{25}-A_{35}+$ &c.$)$. If these higher harmonics are absent, or if they are thus taken into account, the values obtained by the simple averaging will be true, and will contain absolutely no constituent due to any other harmonic, odd or even, of either higher or lower order.

2. The rationale of the process will be evident from a consideration of the figures which follow. If in Fig. 1 the base-

line 0 to 2π represent one period, then the curve represents a fifth sine harmonic; and mere inspection shows that its amplitude will be $\frac{1}{10}$ of the sum of the 10 equidistant ordinates *if taken alternately positively and negatively* as to their values as they stand. Fig. 2 shows a fundamental sine curve of the same period, with ordinates raised at the same points on the base line; while Fig. 3 exhibits the curve compounded out of Figs. 1 and 2. The heights of the 10 consecutive ordinates of Fig. 3 are, in millimetres,

$$6, \quad 5, \quad 13, \quad 5, \quad 6, \; -6, \; -5, \; -13, \; -5, \; -6.$$

If we reverse the signs of the alternate ordinates we have

$$6, \; -5, \; +13, \; -5, \; +6, \; +6, \; -5, \; +13, \; -5, \; +6,$$

and the sum of these is 30, and the mean value is 3. Hence we learn that $A_5=3$. The ordinate at 90° is 13, and is the sum of A_5 and A_1; hence we learn that $A_1=10$, and the equation to the curve of Fig. 3 is $y=10 \sin x+3 \sin 5x$.

FIG. 4.

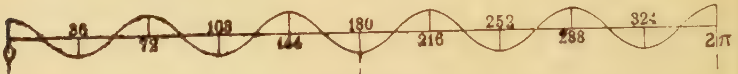


FIG. 5.

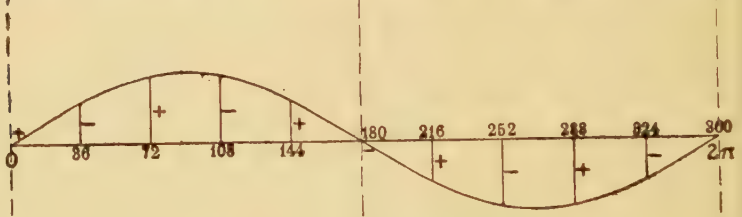
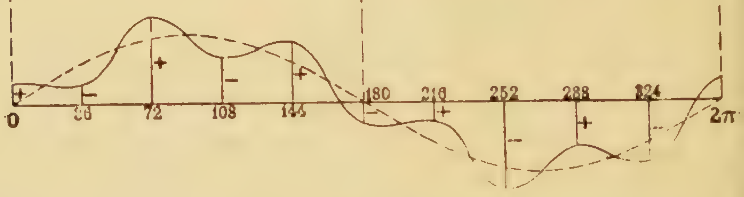


FIG. 6.



Similarly, Fig. 4 depicts a fifth cosine harmonic, and Fig. 6 gives this compounded with the fundamental sine curve. To ascertain the cosine component of the fifth order we must measure 10 ordinates, at every 36°, beginning at zero. These ten measured ordinates are

$$3, \quad 2{\cdot}9, \quad 12{\cdot}5, \quad 6{\cdot}5, \quad 8{\cdot}9, \; -3, \; -2{\cdot}9, \; -12{\cdot}5, \; -6{\cdot}5, \; -8{\cdot}9.$$

Changing the signs of alternate terms we obtain

$$3, \; -2{\cdot}9, \; +12{\cdot}5, \; -6{\cdot}5, \; +8{\cdot}9, \; +3, \; -2{\cdot}9, \; +12{\cdot}5, \; -6{\cdot}5, \; +8{\cdot}9.$$

The mean of these is 3. Hence the equation to the curve is

$$y=10 \sin x+3 \cos 5x.$$

It will be remarked that, in these two illustrations of the method, it would have sufficed to average over half a period. But had we taken an example containing any harmonic of an even order it would not have been so. For generality, it is necessary to average over a whole period.

Had we wished to ascertain the third harmonic constituents we should have had to use six equidistant ordinates at 60° apart. For the nth harmonic $2n$ ordinates must be taken π/n apart.

In thus averaging from so few ordinates the question arises: Do we really eliminate constituents of other orders? In the case of cosine constituents it is obvious from the mere symmetry of Fig. 5 that the equidistant ordinates of the fundamental, if taken alternately positively and negatively, add up to zero. The same is true of Fig. 2 if taken over the whole period, though not obvious (yet true) for each half period. For harmonics of other orders it is also true, except for the cases already mentioned of harmonics of a frequency which is an odd multiple of the frequency of the harmonic which is sought.

3. To prove that this same process of adding, with alternate signs, the ordinates spaced out at intervals of half a period of the harmonic sought, will, when averaged, give the amplitude of that harmonic, consider the sum S of the series of $2n$ terms.

$$S=B_n \cos 0-B_n \cos \pi+B_n \cos 2\pi-B_n \cos 3\pi \dots \dots -B_n \cos (2n-1)\pi;$$

the value of which is obviously $2nB_n$. Hence B_n is the value obtained by summing up the $2n$ ordinates reckoned alternately positively and negatively, and dividing by $2n$.

4. The generality of the proposition that the fundamental is eliminated from the sum is easily demonstrated. For generality let us assume that the first ordinate is taken at an arbitrary angle, a, and that the succeeding ordinates are spaced apart at successive distances β, where β is equal to π/n, for the purpose of finding the amplitude of the nth harmonic. When we are finding sine constituents $a=b\beta=\pi/2n$. When we are finding cosine constituents we must take $a=0$. Then the sum required (for sines) is the following, to $2n$ terms:—

$$S=\sin a-\sin (a+\beta)+\sin (a+2\beta)-\sin (a+3\beta) + \dots \dots -\sin \{a+(2n-1)\beta\}.$$

This is equivalent to

$$S=\sin \alpha+\sin \{\alpha+(\beta+\pi)\}+\sin \{\alpha+2(\beta+\pi)\}$$
$$+\sin \{\alpha+3(\beta+\pi)\}+\ldots\ldots+\sin \{\alpha+(2n-1)(\beta+\pi)\}.$$

Then, by a known theorem, the sum of this series is

$$S=\frac{\sin 2n\frac{\beta+\pi}{2}}{\sin \frac{\beta+\pi}{2}} \sin\left\{\alpha+(2n-1)\frac{\beta+\pi}{2}\right\}.$$

Inserting the values of α and β, we obtain

$$S=\frac{\sin (n+1)\pi}{\sin \frac{n+1}{n}\pi} . \sin \frac{\pi}{2n}\{1+(2n-1)(n+1)\}.$$

This expression is of zero value for all integral values of n above unity, since $\sin (n+1)\pi$ vanishes, whether n be odd or even.

The corresponding series of cosines, with $\alpha=0$, has for its sum

$$S=\frac{\sin (n+1)\pi}{\sin \frac{n+1}{n}\pi} . \cos \frac{\pi}{2n}(2n-1)(n+1),$$

which likewise vanishes for all integral values of n above unity.

5. In dealing with those higher harmonics, the order of which is an odd multiple of the order that is being sought, we find that the sum does not vanish. For consider the ordinate y_1 at the position $\theta=\frac{1}{2}\frac{\pi}{n}$, which is the first position for finding the sine constituent of the nth order.

$$y_1=A_1 \sin \theta+A_2 \sin 2\theta+A_3 \sin 3\theta+, \&c.$$

$$=A_1 \sin \frac{1}{2}\frac{\pi}{n}+A_2 \sin \frac{2}{2}\frac{\pi}{n}+A \sin \frac{3}{2}\frac{\pi}{n}+, \&c.$$

$=\Sigma A_m \sin \frac{m}{n}\frac{\pi}{2}$; where m has the successive values 1, 2, 3, &c.

$$=\Sigma A \sin \frac{m}{n}\frac{\pi}{2}.$$

If in any term of this series m/n is an even integer, then that term will vanish. If m/n is an odd integer, that term will have the value $\pm A_m$, the sign being $+$ if m/n is 1, 5, 9, &c., and $-$ if m/n is 3, 7, 11, &c. If m/n is not an integer, that one term will not vanish *per se*. But if we regard not simply y_1 but the whole series of ordinates, y_1, y_2, y_3, &c., spaced out at successive intervals of 2θ, over the whole period, then the sum, taken with alternate positive and negative signs, will be cumulative for odd integral values of m/n, and will have the value of nA_m, and will vanish for all values of m/n that are not integral. It is for this reason that the process of averaging for any harmonic requires either that the higher harmonics, the ordinal numbers of which are odd multiples of its ordinal number, be absent, or that they shall be separated evaluated, and so eliminated. Hence, in the schedules which are arranged below for carrying out the process, the higher harmonics are evaluated before the lower ones, and the fundamental last. A caution is necessary that the method may fail to give correct results if higher harmonics are present and are not thus eliminated.*

6. For curves of alternating electric currents and alternating voltages in which odd harmonics only are present, it suffices to average the selected ordinates over a half-period only. But for curves of valve-motions and other periodic quantities which contain even harmonics, the ordinates must be averaged over the whole period.

This process has been found to be specially well adapted to particular cases. For example, in dealing with the voltage curve of an alternator having in its armature six slots per pole, where it is known that probably the only harmonics present are the 5th, 7th, 11th and 13th, these can be evaluated directly without having to work out any others of the series.

Three different schedules for facilitating the process are given below, the first of them being a simple form suitable for valve-motion curves. The second is for odd harmonics only up to the 9th order. The third for odd and even harmonics up to the 7th. An example of an analysis by the first schedule is appended.

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* For Cognate processes see Strachey, "Proc. Roy. Soc." XLII., p. 67 1887; Darwin, "Brit. Assoc. Report," 1883, p. 49; Burkhardt, "Jahresbericht der deutschen Mathematiker, Vereinigung," 1901-2-3, X., pp. 279-288, &c. Fischer-Hinnen, "Elektrotechnik und Maschinenbau," XXVII., 1909, p. 335.

SCHEDULE I.

(Simple schedule for valve-gear diagrams, &c.)

HARMONIC ANALYSIS BY SIMPLE AVERAGING OF SELECTED ORDINATES.

(Harmonics above those of the 3rd order assumed to be absent.)

To find A_1, A_2, A_3, B_1, B_2, B_3.

Divide the base-line of one period to read off in degrees, and measure off the ordinates as required :—

$$A_3=\tfrac{1}{6}(y_{30^\circ}-y_{90^\circ}+y_{150^\circ}-y_{210^\circ}+y_{720^\circ}-y_{330^\circ})\ ;$$
$$B_3=\tfrac{1}{6}(y_{0^\circ}-y_{60^\circ}+y_{120^\circ}-y_{180^\circ}+y_{240^\circ}-y_{300^\circ})\ ;$$
$$A_2=\tfrac{1}{4}(y_{45^\circ}-y_{135^\circ}+y_{225^\circ}-y_{315^\circ})\ ;$$
$$B_2=\tfrac{1}{4}(y_{0^\circ}-y_{90^\circ}+y_{180^\circ}-y_{270^\circ})\ ;$$
$$A_1=\tfrac{1}{2}(y_{90^\circ}-y_{270^\circ})+A_3\ ;$$
$$B_1=\tfrac{1}{2}(y_{0^\circ}-y_{180^\circ})-B_3.$$

The mean ordinate A_0 can be found thus—

$$A_0=y_0-B_1-B_2-B_3.$$

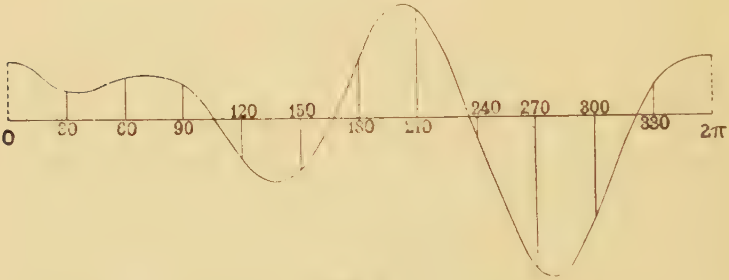

FIG. 7.

Example of the use of Schedule I. Analysis of the curve given in Fig. 7.

This curve was calculated from the equation $y=6\sin\theta+8\sin 2\theta+10\cos 2\theta-10\sin 3\theta$, and plotted to a scale of $\frac{1}{10}$th of an inch as unit. Another person then took off the ordinates from the curve and found them as follows :—

$y_0=10{\cdot}0$
$y_{30}=5{\cdot}0$
$y_{45}=5{\cdot}3$
$y_{60}=7{\cdot}2$
$y_{90}=6{\cdot}0$
$y_{120}=-6{\cdot}8$
$y_{135}=-10{\cdot}9$
$y_{150}=-8{\cdot}9$
$y_{180}=10{\cdot}0$
$y_{210}=18{\cdot}5$
$y_{225}=10{\cdot}7$
$y_{240}=-3{\cdot}4$
$y_{270}=-25{\cdot}9$
$y_{300}=-17{\cdot}3$
$y_{315}=-4{\cdot}7$
$y_{330}=5{\cdot}1$

Inserting these values in the schedule gave the following results :—

$$A_3=\tfrac{1}{6}(5{\cdot}0-6{\cdot}0-8{\cdot}91-8{\cdot}5-25{\cdot}9-5{\cdot}1) = -9{\cdot}90\ ;$$
$$B_3=\tfrac{1}{6}(10{\cdot}0-7{\cdot}2-6{\cdot}8-10{\cdot}0-3{\cdot}4+17{\cdot}3) = -0{\cdot}013\ ;$$
$$A_2=\tfrac{1}{4}(5{\cdot}3+10{\cdot}8+10{\cdot}7+4{\cdot}7) = 7{\cdot}87\ ;$$
$$B_2=\tfrac{1}{4}(10{\cdot}0-6{\cdot}0+10{\cdot}0+25{\cdot}9) = 9{\cdot}98\ ;$$
$$A_1=\tfrac{1}{2}(6{\cdot}0+25{\cdot}9)-9{\cdot}90 = 6{\cdot}05\ ;$$
$$B_1=\tfrac{1}{2}(10{\cdot}0-10{\cdot}0)+0{\cdot}013 = 0{\cdot}013\ ;$$

The slight discrepancies, when compared with the original amplitudes used to calculate the curve, arise from the unavoidable errors in plotting or in reading the values of the ordinates of the curve as plotted. They do not arise from any want of accuracy in the method itself.

SCHEDULE II.

HARMONIC ANALYSIS BY AVERAGING OF SELECTED ORDINATES.

Odd Harmonics only up to the 9th order.

First find the correct base-line, so that the mean ordinate is zero. Then divide the base-line of one period to read in degrees, and measure off ordinates as required in the following formulæ :—

$$A_9=\tfrac{1}{9}(y_{10^\circ}-y_{30^\circ}+y_{50^\circ}-y_{70^\circ}+y_{90^\circ}-y_{110^\circ}+y_{130^\circ}-y_{150^\circ}+y_{170^\circ})\ ;$$
$$B_9=\tfrac{1}{9}(y_{0^\circ}-y_{20^\circ}+y_{40^\circ}-y_{60^\circ}+y_{80^\circ}-y_{100^\circ}+y_{120^\circ}-y_{140^\circ}+y_{160^\circ})\ ;$$
$$A_7=\tfrac{1}{7}(y_{12\cdot84}-y_{38\cdot56}+y_{64\cdot28}-y_{90^\circ}+y_{115\cdot71}-y_{141\cdot43}+y_{167\cdot14})\ ;$$
$$B_7=\tfrac{1}{7}(y_{0^\circ}-y_{25\cdot71}+{}_{51\cdot43}-y_{77\cdot14}+y_{102\cdot85}-y_{128\cdot55}+y_{154\cdot28})\ ;$$
$$A_5=\tfrac{1}{5}(y_{18^\circ}-y_{54^\circ}+y_{90^\circ}-y_{126^\circ}+y_{162^\circ})\ ;$$
$$B_5=\tfrac{1}{5}(y_{0^\circ}-y_{36^\circ}+y_{72^\circ}-y_{108^\circ}+y_{144^\circ})\ ;$$
$$A_3=\tfrac{1}{3}(y_{30^\circ}-y_{90^\circ}+y_{150^\circ})+A_9\ ;$$
$$B_3=\tfrac{1}{3}(y_{0^\circ}-y_{60^\circ}+y_{120^\circ})-B_9\ ;$$
$$A_1=y_{90^\circ}+A_3-A_5+A_7-A_9\ ;$$
$$B_1=y_{0^\circ}-B_3-B_5-B_7-B_9.$$

SCHEDULE III.

HARMONIC ANALYSIS BY AVERAGING OF SELECTED ORDINATES.

(Harmonics above the 7th order assumed absent.)

First graduate the base-line of one period to read in degrees and then measure off ordinates as required in following formulæ :—

$$A_7=\tfrac{1}{14}(y_{12\cdot84}-y_{38\cdot56}+y_{64\cdot28}-y_{90}+y_{115\cdot71}-y_{141\cdot43}+y_{167.14}-y_{192\cdot85}+y_{218\cdot56}$$
$$-y_{244\cdot28}+y_{270}-y_{295\cdot71}+y_{321\cdot43}-y_{347\cdot14})\,;$$
$$B_7=\tfrac{1}{14}(y_0-y_{25\cdot71}+y_{51\cdot43}-y_{77\cdot14}+y_{102\cdot85}-y_{128\cdot56}+y_{154\cdot28}-y_{180}+y_{205\cdot71}$$
$$-y_{231\cdot43}+y_{257\cdot14}-y_{282\cdot85}+y_{308\cdot56}-y_{334\cdot28})\,;$$
$$A_6=\tfrac{1}{12}(y_{15}-y_{45}+y_{75}-y_{105}+y_{135}-y_{165}+y_{195}-y_{225}+y_{255}-y_{285}+y_{315}-y_{545})\,;$$
$$B_6=\tfrac{1}{12}(y_0-y_{30}+y_{60}-y_{90}+y_{120}-y_{150}+y_{180}-y_{210}+y_{240}-y_{270}+y_{300}-y_{330})\,;$$
$$A_5=\tfrac{1}{10}(y_{18}-y_{54}+y_{90}-y_{126}+y_{162}-y_{198}+y_{234}-y_{270}+y_{306}-y_{342})\,;$$
$$B_5=\tfrac{1}{10}(y_0-y_{36}+y_{72}-y_{108}+y_{144}-y_{180}+y_{216}-y_{252}+y_{288}+y_{324})\,;$$
$$A_4=\tfrac{1}{8}(y_{22\cdot5}-y_{67\cdot5}+y_{112\cdot5}-y_{157\cdot5}+y_{202\cdot5}-y_{247\cdot5}+y_{292\cdot5}-y_{337\cdot5})\,;$$
$$B_4=\tfrac{1}{8}(y_0-y_{45}+y_{90}-y_{135}+y_{180}-y_{225}+y_{270}-y_{315})\,;$$
$$A_3=\tfrac{1}{6}(y_{30}-y_{90}+y_{150}-y_{210}+y_{270}-y_{330})\,;$$
$$B_3=\tfrac{1}{6}(y_0-y_{60}+y_{120}-y_{180}+y_{240}-y_{300})\,;$$
$$A_2=\tfrac{1}{4}(y_{45}-y_{135}+y_{225}-y_{315})+A_6\,;$$
$$B_2=\tfrac{1}{4}(y_0-y_{90}+y_{180}-y_{270})-B_6\,;$$
$$A_1=\tfrac{1}{2}(y_{90}-y_{270})+A_3-A_5+A_7\,;$$
$$B_1=\tfrac{1}{2}(y_0+y_{180})-B_3-B_5-B_7\,;$$
$$A_0=y_0-B_1-B_2-B_3-B_4-B_5-B_6-B_7.$$

ABSTRACT.

Assume with Fourier that the curve representing any periodic single-valued function of x may be expressed by the harmonic series;

$$y=A_1\sin x+A_2\sin 2x+A_3\sin 3x+\;.\;.\;.\;.\;.$$
$$+B_1\cos x+B_2\cos 2x+B_3\cos 3x.$$

Then to find the coefficient of any term, A_n or B_n, it suffices—subject to a limitation stated below—to measure off on the curve $2n$ equidistant ordinates over one period, that is, spaced at successive intervals apart of π/n. Then, having reversed the sign of every alternate ordinate, the simple algebraic mean of them gives the coefficient sought. For cosine-coefficients the first ordinate must be taken at the origin; while for sine-coefficients the first ordinate must be taken at a point $\frac{1}{2}\frac{\pi}{n}$ from the origin. In symbols this is

$$A_n=\frac{1}{2n}\left\{y_{\frac{1}{2}\pi/n}-y_{\frac{3}{2}\pi/n}+y_{\frac{5}{2}\pi/n}-y_{\frac{7}{2}\pi/n}+\;.\;.\;.\;.\;-y_{\frac{4n-1}{2}\pi/n}\right\};$$
$$B_n=\frac{1}{2n}\left\{y_0\quad-y_{\pi/n}+y_{2\pi/n}-y_{3\pi/n}+\;.\;.\;.\;.\;-y_{(2n-1)\pi/n}\right\}.$$

The limitation is that if, in finding the nth harmonic, there be present also the $3n$th, or the $5n$th, or the $7n$th, &c., these will be included in the average found by the process, and must, if present, be separately determined and eliminated.

The process is much facilitated by the use of templates of transparent celluloid having equispaced vertical lines engraved upon them. They are laid down on the curve, and the values of the selected ordinates are thus readily measured off. For analysis of valve-motions, of alternating-current curves, of tidal observations, and diurnal magnetic variations, the method presents certain advantages, as it requires no multiplication of ordinates by sines or cosines.

DISCUSSION.

The **President** commented upon the extreme ease with which the method adapted itself to the analysis of experimental curves.

Dr. **Russell** remarked that the method can be usefully employed in interpolation as well as in harmonic analysis. In this connection he gave a brief résumé of Gauss's extension of Lagrange's interpolation formula ("Werke," Vol. III., p. 281). He also pointed out certain cases in which Fourier's analysis fails to be of practical use.

Sir. G. H. **Darwin** communicated the following remarks: In 1874 Mr. Archibald Smith published, in the Admiralty Scientific Manual on "Deviations of the Compass," schedules whereby the multiplications in the usually accepted process are simple. This same arrangement has been in use for the harmonic analysis of tides since 1870, and in his "Manual of Tidal Observations," Major Baird gave multiplication tables for the sines of 15, 45, 60, 75 degs., which render this part of the work so easy that avoidance of all multiplication hardly seems necessary. In 1884, in "Tables and Formulæ to Facilitate the Computation of Harmonic Coefficients," General Sir Richard Strachey gave rules which reduce the multiplications still further. In both these schemes Prof. Thompson's principle is used for some of the higher harmonics. Smith's and Strachey's schedules are arranged for 24 hourly values, but they may be equally well used for 12 ordinates. Schedules for the use of any other number of ordinates would have to be prepared specially, and in such cases Prof. Thompson's rules might prove very convenient. I have not worked out the probable errors in the several methods, but it seems likely that they would be smaller in Smith's method than in any other. Examples of both these schedules will be found in Vol. I. of my "Collected Papers," or in the report to the Brit. Assoc. on Harmonic Analysis (1883), and to article "Tides," in the "Admiralty Scientifici Manual" (1886).

XXXII. *Short Table of Circular and Hyperbolic Functions for Complex Values of the Argument.* *By* A. JOHNSTONE, *B.Sc.*

COMMUNICATED BY PROF. C. H. LEES, F.R.S. RECEIVED MAY 12, 1911.

IN view of the recent prominence given to the circular and hyperbolic functions for complex values of the argument by the use of them in calculations regarding telegraphic and telephonic circuits, I have, at the suggestion of Prof. Lees, calculated out the following table in which the values of the cosh and sinh functions will be found.

In order to make the table more generally available the following formulæ, connecting the values of the functions for different values of the argument are given :—

Values of Cosh $(x+iy)=R+iU$.

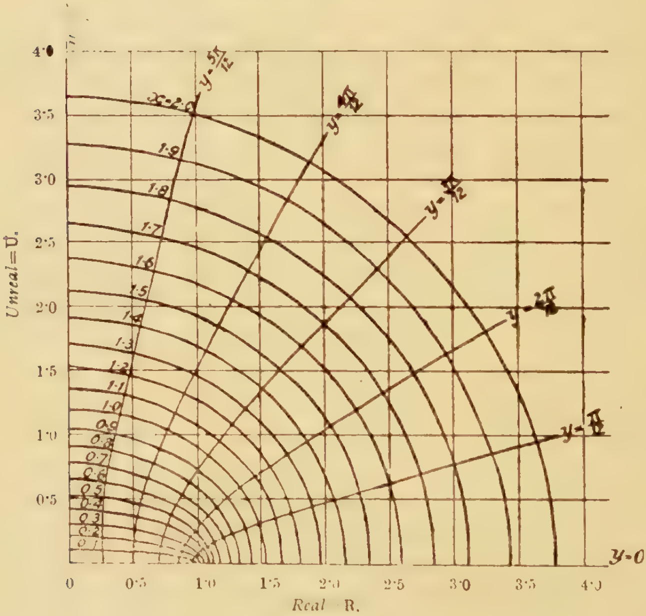

FIG. 1.

1. $\text{Cosh} - (x+iy) = \cosh (x+iy)$.
2. $\text{Sinh} - (x+iy) = -\sinh (x+iy)$.

Cosh $(x+iy)$.

Values of x.	Values of y. 0		$15° = \frac{\pi}{12} = 0·2618$		$30° = \frac{2\pi}{12} = 0·5236$		$45° = \frac{3\pi}{12} = 0·7854$		$60° = \frac{4\pi}{12} = 1·0472$		$75° = \frac{5\pi}{12} = 1·3090$		$90° = \frac{6\pi}{12} = 1·5708$	
	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal
	R	U	R	U	R	U	R	U	R	U	R	U	R	U
0·0	1·0000	0	0·9659	0	0·8660	0	0·7071	0	0·5000	0	0·2588	0	0	0
0·1	1·0050	...	0·9708	0·0259	0·8704	0·0501	0·7106	0·0709	0·5025	0·0868	0·2601	0·0968	...	0·1002
0·2	1·0201	...	0·9853	0·0521	0·8834	0·1007	0·7213	0·1423	0·5100	0·1743	0·2640	0·1944	...	0·2013
0·3	1·0453	...	1·0097	0·0788	0·9053	0·1523	0·7391	0·2153	0·5227	0·2637	0·2705	0·2941	...	0·3045
0·4	1·0811	...	1·0443	0·1063	0·9363	0·2054	0·7645	0·2905	0·5405	0·3558	0·2798	0·3968	...	0·4108
0·5	1·1276	...	1·0892	0·1349	0·9765	0·2606	0·7973	0·3685	0·5638	0·4513	0·2918	0·5033	...	0·5211
0·6	1·1855	...	1·1451	0·1648	1·0267	0·3183	0·8383	0·4502	0·5927	0·5514	0·3068	0·6150	...	0·6367
0·7	1·2552	...	1·2124	0·1963	1·0870	0·3793	0·8876	0·5364	0·6276	0·6570	0·3249	0·7328	...	0·7586
0·8	1·3374	...	1·2918	0·2299	1·1582	0·4441	0·9457	0·6280	0·6687	0·7691	0·3461	0·8578	...	0·8881
0·9	1·4331	...	1·3843	0·2657	1·2411	0·5133	1·0134	0·7259	0·7165	0·8890	0·3709	0·9915	...	1·0265
1·0	1·5431	...	1·4905	0·3042	1·3364	0·5876	1·0911	0·8310	0·7715	1·0178	0·3994	1·1352	...	1·1752
1·1	1·6685	...	1·6116	0·3457	1·4450	0·6678	1·1798	0·9445	0·8343	1·1568	0·4318	1·2902	...	1·3357
1·2	1·8107	...	1·7490	0·3907	1·5681	0·7547	1·2804	1·0674	0·9053	1·3073	0·4686	1·4581	...	1·5095
1·3	1·9709	...	1·9037	0·4396	1·7068	0·8492	1·3936	1·2010	0·9855	1·4709	0·5101	1·6405	...	1·6984
1·4	2·1509	...	2·0776	0·4929	1·8627	0·9522	1·5209	1·3465	1·0755	1·6492	0·5567	1·8394	...	1·9043
1·5	2·3524	...	2·2722	0·5511	2·0372	1·0646	1·6634	1·5056	1·1762	1·8440	0·6089	2·0567	...	2·1293
1·6	2·5775	...	2·4897	0·6149	2·2322	1·1878	1·8226	1·6798	1·2887	2·0573	0·6671	2·2947	...	2·3756
1·7	2·8283	...	2·7319	0·6847	2·4494	1·3228	1·9999	1·8707	1·4142	2·2912	0·7320	2·5555	...	2·6456
1·8	3·1075	...	3·0016	0·7615	2·6912	1·4711	2·1973	2·0804	1·5537	2·5480	0·8043	2·8419	...	2·9422
1·9	3·4177	...	3·3012	0·8459	2·9598	1·6341	2·4167	2·3110	1·7089	2·8303	0·8846	3·1568	...	3·2682
2·0	3·7622	...	3·6340	0·9387	3·2582	1·8134	2·6603	2·5646	1·8811	3·1410	0·9737	3·5033	...	3·6269
2·1	4·1443	...	4·0031	1·0409	3·5891	2·0109	2·9305	2·8439	2·0722	3·4831	1·0726	3·8849	...	4·0219
2·2	4·5679	...	4·4123	1·1536	3·9559	2·2286	3·2300	3·1516	2·2840	3·8600	1·1823	4·3052	...	4·4571
2·3	5·0372	...	4·8656	1·2778	4·3623	2·4685	3·5618	3·4910	2·5186	4·2756	1·3037	4·7688	...	4·9370
2·4	5·5570	...	5·3677	1·4148	4·8125	2·7331	3·9294	3·8652	2·7785	4·7339	1·4383	5·2799	...	5·4662
	U′	R′	U′	R′	U′	R′	U′	R′	U′	R′	U′	R′	U′	R′
	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real
	$90° = \frac{6\pi}{12} = 1·5708$		$75° = \frac{5\pi}{12} = 1·3090$		$60° = \frac{4\pi}{12} = 1·0472$		$45° = \frac{3\pi}{12} = 0·7854$		$30° = \frac{2\pi}{12} = 0·5236$		$15° = \frac{\pi}{12} = 0·2618$		0	

Values of x.	0		$15° = \frac{\pi}{12} = 0·2618$		$30° = \frac{2\pi}{12} = 0·5236$		$45° = \frac{3\pi}{12} = 0·7854$		$60° = \frac{4\pi}{12} = 1·0472$		$75° = \frac{5\pi}{12} = 1·3090$		$90° = \frac{6\pi}{12} = 1·5708$	
	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal
	R	U	R	U	R	U	R	U	R	U	R	U	R	U
2·5	6·1323	0	5·9233	1·5659	5·3107	3·0251	4·3362	4·2781	3·0661	5·2396	1·5872	5·8440	0	6·0502
2·6	6·7690	...	6·5384	1·7327	5·8621	3·3474	4·7864	4·7339	3·3845	5·7978	1·7519	6·4666	...	6·6947
2·7	7·4735	...	7·2188	1·9169	6·4722	3·7031	5·2846	5·2370	3·7367	6·4140	1·9343	7·1539	...	7·4063
2·8	8·2527	...	7·9715	2·1202	7·1470	4·0960	5·8355	5·7925	4·1264	7·0944	2·1360	7·9128	...	8·1919
2·9	9·1146	...	8·8040	2·3448	7·8935	4·5298	6·4450	6·4061	4·5573	7·8458	2·3590	8·7509	...	9·0596
3·0	10·068	...	9·7247	2·5928	8·7189	5·0089	7·1189	7·0837	5·0339	8·6758	2·6057	9·6765	...	10·018
3·2	12·287	...	11·868	3·1695	10·641	6·1230	8·6879	8·6592	6·1433	10·605	3·1800	11·829	...	12·246
3·4	14·999	...	14·488	3·8733	12·989	7·4827	10·606	10·582	7·4994	12·960	3·8819	14·455	...	14·965
3·6	18·313	...	17·689	4·7326	15·859	9·1428	12·949	12·930	9·1564	15·836	4·7397	17·662	...	18·286
3·8	22·362	...	21·600	5·7819	19·366	11·170	15·812	15·796	11·181	19·346	5·7877	21·578	...	22·339
4·0	27·308	...	26·378	7·0631	23·650	13·645	19·310	19·297	13·654	23·634	7·679	26·360	...	27·290
4·5	45·014	...	43·480	11·648	38·983	22·502	31·830	31·822	22·507	38·974	11·651	43·470	...	45·003
5·0	74·210	...	71·681	19·205	64·268	37·102	52·474	52·470	37·105	64·262	19·207	71·675	...	74·203
5·5	122·35	...	118·18	31·665	105·96	61·172	86·513	86·510	61·174	105·95	31·666	118·18	...	122·34
6·0	201·72	...	194·84	52·207	174·69	100·86	142·63	142·63	100·86	174·69	52·208	194·84	...	201·71
6·5	332·57	...	321·24	86·076	288·01	166·29	235·16	235·16	166·29	288·01	86·076	321·24	...	332·57
7·0	548·32	...	529·63	141·91	474·86	274·16	387·72	387·72	274·16	474·86	141·91	529·63	...	548·32
8·0	1490·5	...	1439·7	385·76	1290·8	745·24	1053·9	1053·9	745·24	1290·8	385·76	1439·7	...	1490·5
9·0	4051·5	...	3913·5	1048·6	3508·7	2025·8	2864·9	2864·9	2025·8	3508·7	1048·6	3913·5	...	4051·5
10·0	11012	...	10638	2850·4	9537·7	5506·6	7787·5	7787·5	5506·6	9537·7	2850·4	10638	...	11013
Approaches x Large	$e^x/2$ U′	... R′	$0·9659e^x/2$ U′	$0·2588e^x/2$ R′	$0·8660e^x/2$ U′	$0·5000e^x/2$ R′	$0·7071e^x/2$ U′	$0·7071e^x/2$ R′	$0·5000e^x/2$ U′	$0·8660e^x/2$ R′	$0·2588e^x/2$ U′	$0·9659e^x/2$ R′	... U′	$e^x/2$ R′
	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real	Unreal	Real
	$90° = \frac{6\pi}{12} = 1·5708$		$75° = \frac{5\pi}{12} = 1·3090$		$60° = \frac{4\pi}{12} = 1·0472$		$45° = \frac{3\pi}{12} = 0·7854$		$30° = \frac{2\pi}{12} = 0·5236$		$15° = \frac{\pi}{12} = 0·2618$		0	
	VALUES OF y.													

To face p. 345.

Sinh $(x+iy)$.

3. $\text{Cosh}\,(x \pm iy) = R' \pm iU.$
4. $\text{Sinh}\,(x \pm iy) = R \pm iU'.$
5. $\text{Cosh}\,(x+iy+2n\pi i) = \cosh\,(x+iy),$ n integral.
6. $\text{Sinh}\,(x+iy+2n\pi i) = \sinh\,(x+iy).$
7. $\text{Cosh}\,(x+iy+\pi i) = -\cosh\,(x+iy).$
8. $\text{Sinh}\,(x+iy+\pi i) = -\sinh\,(x+iy).$
9. $\text{Cosh}\left(x+iy+\frac{\pi}{2}i\right) = i\sinh\,(x+iy).$
10. $\text{Sinh}\left(x+iy+\frac{\pi}{2}i\right) = i\cosh\,(x+iy).$
11. $\text{Cos}\,(x+iy) = \cosh\,i(x+iy) = \cosh\,(ix-y) = \cosh\,(y-ix).$
12. $\text{Sin}\,(x+iy) = -\sinh\,i(x+iy) = -i\sinh\,(ix-y) = i\sinh\,(y-ix).$

EXPLANATION OF THE TABLE.

To find the value of $\frac{\cosh}{\sinh}(x+iy)$, reduce y to angular measure by multiplying by $\frac{180}{\pi}$. Subtract from its value any multiple of 360 deg. (*see* equations 5 or 6 above). If the difference is less than 90 deg., use the table $\frac{\cosh}{\sinh}(x+iy)$. If it is greater than 90 deg. but less than 180 deg., subtract 90 deg. and use the table $\frac{\sinh}{\cosh}(x+iy)$ (*see* equations 9 or 10 above). If it is greater than 180 deg., subtract 180 deg. and proceed as before (*see* equations 7 or 8 above).

Interpolation for values of x greater than 1 will be facilitated by Fig. 1.

By the aid of equations 11 and 12 above $\frac{\cos}{\sin}(x+iy)$ may be found in the same way, the value of x being expressed in angular measure.

XXXIII. *On the Measurement of Contact Differences of Potential. By Prof.* A. Anderson, *M.A., and* J. E. Bowen, *M.A.*

Received May 15, 1911.

The following methods of determining the Volta, or contact, differences of potential of pairs of different metals are the outcome of the overcoming of some difficulties that arose in working with a Dolezalek quadrant electrometer. The values obtained, though correctly measured to the third decimal place, lay no claim to any scientific importance, insamuch as they, in common with results obtained by other methods, depend so greatly on the state of the surfaces at the time of measurement. In several cases the changes with time after the metals were cleaned were considerable, and very notably so in the case of aluminium.

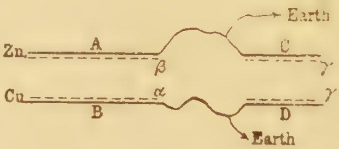

Fig. 1.

The following statement of the theory usually adopted will make the methods clear :—

The layer of air in contact with a metal in chemical equilibrium with it is at a potential different from that of the metal the difference varying for different metals and for different states of the surface of the same metal.

In Fig. 1, A represents a plate of zinc and B a plate of copper. C and D represent the two pairs of quadrants of the electrometer. If, as in the figure, A, B, C and D are all connected to earth, the potential of the layer of air in contact with A will be, say, β, that of the layer of air in contact with B, α, and that of the air in contact with both pairs of quadrants, γ. There is thus a field of force in the air between A and B, but no field of force in the air between the pairs of quadrants.

If, now, A and C are insulated, and an ionising source be employed to destroy the field between A and B, we shall have, when equilibrium is attained, the state of affairs represented in Fig. 2.

The flow of negative electrons and positive atoms has reduced the potential of A to $-(\beta-\alpha)$. The potential of the air in contact with one pair of quadrants is $-(\beta-\alpha)+\gamma$, and that of the air in contact with the other, γ.

The electrometer then measures $\beta-\alpha$, which is what is known as the contact difference of potential. This is the first, or deflection, method.

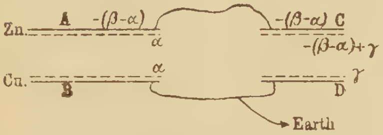

FIG. 2.

The second method is a null, or compensation, method, and will be easily understood from Fig. 3.

The copper plate B is insulated from its pair of quadrants which are kept earthed, the air between A and B ionised, and a positive potential, V, put on to the copper plate, such that, when the earth connection of the zinc plate and its pair of quadrants are broken, there is no deflection of the electrometer needle. We must then have

$$V+\alpha=\beta, \text{ or } V=\beta-\alpha.$$

The apparatus and its arrangement will be readily understood from Fig. 4.

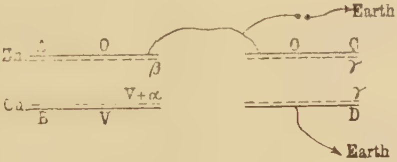

FIG. 3.

E is a copper cylinder resting on a block of paraffin, P, and furnished with a copper lid, L. It was, in the first method, earthed, but in the second method it was insulated and potentials applied to it by means of a Nalder's potentiometer reading to one-thousandth of a volt. B is a copper plate resting on the bottom of the cylinder. This plate is unnecessary, as the bottom of the cylinder will serve the same purpose. A is a plate of zinc or other metal, the contact P.D. between which and copper is to be found. It is supported by a ring of paraffin wax within which, resting on the plate B, was the ionising agent, a sealed glass tube containing radium bromide. The copper wire connecting A to the electrometer passed through an earthed copper tube, T, filled with paraffin, this tube starting from within another copper tube, S, forming part of the cylinder, and ending within the earthed metallic cage containing the electrometer. H is a mercury cup used to break connection between A and the earth. The deflection of the electrometer was 405 mm. for 1 volt.

The results obtained by the two methods agreed very closely, and did not differ by more than two or three units in the third decimal place.

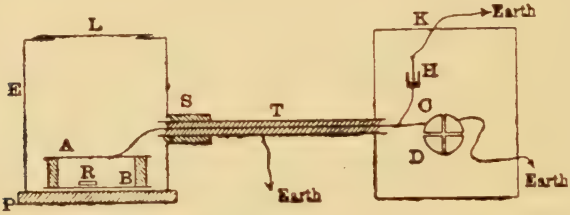

Fig. 4.

To test the addition law a zinc cylinder was used instead of a copper one. The tube T was zinc, and it was found necessary to use a zinc strip instead of a copper wire to connect A with the electrometer.

The following are the results obtained in an experiment in which the second method was used. The metals were cleaned and left aside for about two hours when the measurements were rapidly made. The first column gives the contact differences of potential between copper and nine other metals.

The second column gives the same P.D. calculated from observations made when the zinc cylinder was used, assuming the truth of the addition law.

	Volts.	Volts.
Al	0·824	0·831
Zn	0·762	0·763
Pb	0·550	0·549
Sn	0·472	0·476
Fe	0·194	0·191
Brass	0·154	0·152
Au	−0·115	−0·113
Ag	−0·132	−0·135
Pt	−0·200	−0·198

POSTSCRIPT.

[MAY 31, 1911.]

The results obtained by the two methods were carefully compared and very many measurements were made. At first, the results obtained by the deflection method were always less than those obtained by the null method, but, after some difficulties connected principally with leakage had been overcome, the results were, in all cases, practically identical. All the metals were freshly scratched before measurements were made except the gold, silver and platinum, which were carefully cleaned to remove grease. After fresh scratching the P.D. changed slightly in all cases, but soon came to a practically steady state. In the case of copper-lead, the P.D. increased; in that of copper-aluminium the change, which was a decrease, was more marked. There is no doubt that both of these changes were due principally to the lead and aluminium themselves.

In an experiment recently made the P.D. between a freshly scratched aluminium plate and a copper plate which had been left untouched for over a fortnight was measured. At first the P.D. in volts, copper being positive to aluminium, was 1·062, in 5 minutes it decreased to 1·008, in 10 minutes to 0·984, in 25 minutes to 0·952, in 50 minutes to 0·94, and in 60 minutes to 0·938, after which the rate of change was very slow.

The copper plate was then compared with another copper plate freshly scratched, when the former was found to be positive to the latter, the difference being 0·12, which diminished very slowly, the change being 0·004 in 30 minutes.

It will be seen that the deflection method is particularly well adapted for showing alterations in the contact difference due to lapse of time, or to other changes in the conditions which may be controlled.

ADDENDUM.

[RECEIVED JUNE 7, 1911.]

An opportunity of applying the deflection method presented itself quite recently. A Paper published in "Comptes Rendus" (March 13, 1911) gives some results of a research by M. de Broglieon the lowering of the apparent contact differences of potential by the removal of the adhering layers of moisture. The author concludes that almost the whole of the contact

P.D.s is due to layers of moisture on the plates. We thought it might be worth while to test this by the deflection method.

A zinc plate was insulated inside the copper cylinder, whose capacity is about 1,500 c.c., the surfaces of both metals having been thoroughly scratched with emery paper. The contact potential shown was 0·734 volt. A current of air, dried by being passed through strong sulphuric acid and calcium chloride, and heated during its passage through a glass tube raised to a high temperature, was then pumped through the vessel, which had previously been made as air-tight as possible. The pump was worked for four hours.

The P.D. rose gradually for about an hour, when it stood at 0·796 volt. Part of this, however, was certainly due to a frictional negative charge produced on the zinc plate by the current of dry air flowing past it. When the pump was stopped for a minute the deflection diminished by about eight divisions, corresponding to a decrease of 0·019 volt. The apparent increase of P.D. due to the frictional charge was greater the higher the temperature to which the air was raised, and the more vigorously the pump was worked. The P.D. may also have been influenced by the rise in temperature of the plates, but this effect, if it existed, was small.

During the second hour the P.D. gradually fell to 0·767 volt, and remained at this, with exceedingly small up and down variations, for the remaining two hours, so that it could not be said with certainty that for these two hours there was a change of one-thousandth of a volt. The remarkable thing about it was its steadiness.

At the end of the four hours the pump was stopped, when the P.D. went down quickly to, and remained at, 0·722 volt. The result of four hours' continuous drying was, therefore, a fall of 0·012 volt. If the P.D. can be brought down to a few hundredths of a volt a much more effective method of drying than that used in this experiment would be necessary. Indeed, the experiment gave one the impression that the current of drying air, so far from having the effect of lowering the P.D., actually kept it steady.

It must be noted that M. de Broglie's plates were heated to temperatures varying between 100°C. and 400°C., and then allowed to cool in an enclosure 1 cubic meter in volume containing air kept dry by chloride of calcium.

We are engaged in further experiments with a view to a more complete test of M. de Broglie's result.

ABSTRACT.

The Paper describes two methods of measuring the contact differences of potential of pairs of metals. The first, or deflection, method depends on the property which a radioactive source has of destroying a field of electrostatic force in air, and the second, or null, method on the possibility of determining by means of such a source whether such a field exists between two plates at zero potential. Measurements were made on 10 different metals, and it was found that both methods gave practically the same results, provided that the time which was allowed to elapse between the two measurements was sufficiently small. The addition law was verified.

XXXIV. *The Capacity Coefficients of Spherical Electrodes.* By ALEXANDER RUSSELL, *M.A., D.Sc., M.I.E.E.*

RECEIVED MAY 23, 1911. READ JUNE 30, 1911.

IN practical electrical work a knowledge of the values of the capacity coefficients of spheres is sometimes very useful. Considering the importance of these coefficients it is remarkable that in no book with which the author is acquainted is there a numerical example given of their values when the spheres are unequal. The probable explanation is that physicists have found the ordinary formulæ unsuitable for computation. The author, having recently had to calculate their values in many cases, has been led to modify the old formulæ and to invent new ones with the object of diminishing the numerical labour involved.

The formulæ given below solve the problem completely. The labour entailed in computing by their means begin, even in the worst cases, very little greater than that required to find the angles of a triangle when the sides are given.

Maxwell's Formulæ.

Let the radii of the two spherical conductors be a and b respectively and let c be the distance between their centres. If the charges and potentials of the spheres be q_1, q_2 and v_1, v_2 respectively, we have

$$q_1=k_{1,1}v_1+k_{1,2}v_2 \text{ and } q_2=k_{2,2}v_2+k_{1,2}v_1, \quad . \quad . \quad . \quad (1)$$

where $k_{1,1}$, $k_{1,2}$ and $k_{2,2}$ are the capacity coefficients of the two spheres, the values of which have to be computed from the known values of a, b and c. Maxwell's solution,* slightly modified so as to make it more symmetrical, is given by the following equations :—

$$k_{1,1}=\lambda\Sigma\frac{1}{\sinh(\alpha_1+s\omega_1)} \text{ and} -k_{1,2}=\lambda\Sigma\frac{1}{\sinh(s+1)\omega_1}, \quad . \quad (2)$$

where $$4c^2\lambda^2=(c+a+b)(c-a-b)(c+a-b)(c-a+b) \quad . \quad . \quad (3)$$

and $$\sinh\alpha_1=\lambda/a,\ \sinh\omega_1=\lambda c/ab. \quad . \quad . \quad . \quad . \quad (4)$$

The summation is taken from $s=0$ to $s=$infinity, and the value of $k_{2,2}$ is found by writing β_1 for α_1 in the formula for $k_{1,1}$, where $\sinh\beta_1=\lambda/b$. It is easy to show that λ is the radius of the sphere which cuts the two given spheres orthogonally.

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* *Cf.* Maxwell, Vol. I., §173.

When they are close together λ is very small and, therefore, α_1, β_1 and ω_1 are also very small, and so the labour involved in computing by these formulæ is prohibitive.

Kirchhoff's and Poisson's Formulæ.

If in Maxwell's formulæ we write $\varepsilon^{\alpha_1}=1/\xi$, $\varepsilon^{\beta_1}=1/\eta$ and $\varepsilon=1/\alpha$, we get

$$k_{1,1}=2\lambda\left\{\frac{\xi}{1-\xi^2}\quad\frac{\alpha\xi}{1-\alpha^2\xi^2}+\frac{\alpha^2\xi}{1-\alpha^4\xi^2}+\ldots\right\} \quad . \quad . \quad (5)$$

and

$$-k_{1,2}=2\lambda\left\{\frac{\alpha}{1-\alpha^2}+\frac{\alpha^2}{1-\alpha^4}+\frac{\alpha^3}{1-\alpha^6}+\ldots\right\} \quad . \quad . \quad (6)$$

The formula for $k_{2,2}$ is found by writing η for ξ in (5). It may be shown that

$$\xi=(1+\lambda^2/a^2)_{1/2}-\lambda/a\,;\ \alpha=(c\xi-a)/b, \text{ and } \eta=\alpha/\xi. \qquad (7)$$

The last equation follows since $\omega_1=\alpha_1+\beta_1$. The formulæ were practically given in this shape by Kirchhoff.* An admirably concise proof of them by Kelvin's method of images is given in Jeans' "Electricity and Magnetism," p. 192. Jeans also proves that they are equivalent to the definite integral solutions given by Poisson.†

When ξ and η are nearly unity the numerical labour is very heavy. To get over this difficulty Kirchhoff‡ modified the series so as to make them converge more rapidly by means of the following interesting extension of Clausen's theorem—

$$\frac{1}{1-\alpha}+\frac{\beta}{1-\alpha\gamma}+\frac{\beta^2}{1-\alpha\gamma^2}+\ldots=\Sigma a^n\beta^n\gamma^{n^2}\frac{1-\alpha\beta\gamma^{2n}}{(1\alpha\gamma^n)(1-\beta\gamma^n)},$$

the summation being taken from $n=0$ to $n=$infinity.

The methods given below, however, are better.

Barnes's Solution.

In the "Quarterly Journal," No. 138, 1903, p. 155, Barnes shows that the capacity coefficients can be determined in terms of double Gamma-functions. Until these functions are tabulated, however, the solutions given are no help to the physicist.

The Author's Formulæ.

The author has found the following modification of Kirchhoff's formulæ very convenient when making numerical calculations. Let $2\lambda R_n$ denote the sum after the first n terms of the series for $k_{1,1}$. Then

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* "Gesammelte Abhandlungen," p. 78.

† "Memoires de l'Institut," 1812.

‡ "Annalen der Physik," Vol. XXVII., p. 673, 1886.

$$R_n = \frac{\xi\alpha^n}{1-\xi^2\alpha^{2n}} + \frac{\xi\alpha^{n+1}}{1-\xi^2\alpha^{2n+2}} + \ldots .$$

and hence, by grouping the terms differently, we have

$$R_n = \frac{\xi\alpha^n}{1-\alpha} + \frac{(\xi\alpha^n)^3}{1-\alpha^3} + \frac{(\xi\alpha^n)^5}{1-\alpha^5} + \ldots \quad . \quad . \quad . \quad . \qquad (8)$$

We see that when $\xi\alpha^n$ is small the series (8) converges very rapidly.

Similarly, if $2\lambda R'_n$ be the remainder after n terms of the series for $-k_{1,2}$, we have

$$R'_n = \frac{\alpha^{n+1}}{1-\alpha} + \frac{\alpha^{3n+3}}{1-\alpha^3} + \frac{\alpha^{5n+5}}{1-\alpha^5} + \ldots \quad . \quad . \quad . \quad . \qquad (9)$$

Using this method it will be found that we can readily compute the values of the coefficients even when the distance between the spheres is only the hundredth part of the radius of the smaller sphere.

In the particular case when n is zero, the formula for $k_{1,1}$ becomes

$$k_{1,1} = 2\lambda\left\{\frac{\xi}{1-\alpha} + \frac{\xi^3}{1-\alpha^3} + \frac{\xi^5}{1-\alpha^5} + \ldots\right\} \quad . \quad . \qquad (10)$$

Equally simple formulæ for $-k_{1,2}$ and $k_{2,2}$ are obtained by writing α and η respectively for ξ in (10)

When $n=1$, the formulæ become

$$k_{1,1} = \alpha + 2\lambda\xi\left\{\frac{\alpha}{1-\alpha} + \frac{\alpha^3}{1-\alpha^3}\xi^2 + \frac{\alpha}{1-\alpha^5}\xi^4 + \ldots\right\}. \qquad (11)$$

$$\text{and } -k_{1,2} = \frac{ab}{c} + 2\lambda a\left\{\frac{\alpha}{1-\alpha} + \frac{\alpha^3}{1-\alpha^3}\alpha^2 + \frac{\alpha^5}{1-\alpha^5}\alpha^4 + \ldots\right\}. \qquad (12)$$

Let us suppose, for example, that $a=7r$, $b=r$ and $c=10r$. Then, by (3) and (7)

$$\lambda = 2{\cdot}4r,\ \xi = 5/7,\ \eta = 1/5 \text{ and } \alpha = 1/7.$$

Hence, by (11) and (12),

$$k_{1,1} = 7r + \frac{4}{7}r\{1 + 0{\cdot}0089509 + 0{\cdot}0000929 + 0{\cdot}0000009 + \ldots$$

$$= 7{\cdot}5765970r.$$

$$-k_{1,2} = 0{\cdot}7r + (8/70)\{1 + 0{\cdot}0003580 + 0{\cdot}0000001 + \ldots\}$$

$$= 0{\cdot}8143266r.$$

Similarly,

$$k_{2,2} = 1{\cdot}1601124r.$$

As a further example let us suppose that $a=98$, $b=10{\cdot}8$ and $c=a+b+0{\cdot}2=109$. In this case, taking $n=4$, we find that only seven terms of the series for R_4 need to be computed in order to find that

$$k_{1,1}=1{\cdot}2708614a.$$

When the distance between the spheres is, as in the last example, only a small fraction of the radius of the smaller one, the computation of the coefficients by the above method is laborious. In this case it saves labour to use the formulæ* which the author has previously given. The closer the spheres are to one another the simpler these formulæ become. We shall write for the function $F(x)$† previously used, $-\psi(1/x)-\log x$, where $\psi(x)$ is defined by

$$\psi(x)=\frac{\delta}{\delta x}\{\log \Gamma(x)\}.$$

This function has been tabulated by Gauss.‡

The formulæ now become

$$\frac{k_{1,1}}{2\lambda}=-\frac{\psi(\alpha_1/\omega_1)+\log(\omega_1/2)}{2\omega_1}+\frac{\omega_1}{24}\left(\frac{1}{6}-\frac{\alpha_1\beta_1}{\omega_1^2}\right)$$

$$+\frac{7\omega_1^3}{2880}\left(\frac{1}{30}-\frac{\alpha_1^2\beta_1^2}{\omega_1^4}\right)+\ldots\ldots \quad (13)$$

$$-\frac{k_{1,2}}{2\lambda}=\frac{\gamma-\log(\omega_1/2)}{2\omega_1}+\frac{\omega_1}{144}+\frac{7\omega_1^3}{86400}+\ldots \quad (14)$$

$$\frac{k_{2,2}}{\lambda}=\frac{k_{1,1}}{\lambda}-\frac{\pi}{\omega_1}\cot\frac{\pi\alpha_1}{\omega_1}, \quad (15)$$

where $\gamma=0{\cdot}577216\ldots=$Euler's constant. The function $\psi(x)$ is easily computed by the series

$$\psi(x)=-\frac{1}{x}-\gamma+\frac{x}{1+x}+(S_2-1)x-(S_3-1)x^2+(S_4-1)x^3-\ldots, (16)$$

where $S_m=\sum_1^\infty\frac{1}{n^m}$. The values of $S_2, S_3, \ldots$ are given in Dale's "Mathematical Tables," p. 92.

If the distance x between the spheres be not greater than a tenth of the radius of the smaller sphere and an accuracy of the tenth part of 1 per cent. suffices the following formulæ are the best to use :—

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* "Proc." Royal Society, A, Vol. LXXXII., p. 527.

† *Loc. cit.*, p. 525.

‡ Werke, Bd. 3, pp. 161–162. *See also* "Functionentafeln," Jahnke and Emde, p. 30.

$$k_{1.1}=\frac{ab}{a+b}\left\{1+\frac{a^2+b^2-ab}{3ab(a+b)}x\right\}\left[-\psi\left\{\frac{b}{a+b}+\frac{x(a-b)}{3(a+b)^2}\right\}\right.$$
$$\left.+\tfrac{1}{2}\log\frac{2ab}{(a+b)x}+\frac{4(a-b)^2+ab}{36ab(a+b)}x\right], \quad (17)$$

$$-k_{1.2}=\frac{ab}{a+b}\left\{1+\frac{a^2+b^2-ab}{3ab(a+b}x\right\}\left\{\gamma+\tfrac{1}{2}\log\frac{2ab}{(a+b)x}\right.$$
$$\left.+\frac{4(a-b)^2+7ab}{36ab(a+b)}x\right\}. \quad . \quad (18)$$

The value of $k_{2,2}$ is found by interchanging a and b in (17).

Calculating by (17) when $a=98$, $b=10\cdot 8$ and $c=109$, we get $k_{1,1}/a=1\cdot 27070$. The true value found above is $1\cdot 27086$. Similarly we find that $-k_{1.2}/a=0\cdot 2861$ and $k_{2.2}/a=0\cdot 3036$.

Formula for the Difference of the Self-coefficients.

From (15) we get

$$k_{1.2}-k_{2.2}=\frac{\pi\lambda}{\log(1/\alpha)}\cot\left\{\pi\frac{\log(1/\xi)}{\log(1/\alpha)}\right\}. \quad . \quad . \quad (19)$$

This is a very accurate formula when x is small. Even when x is as great as $3ab/(a+b)$ the maximum inaccuracy is only about 1 in 1,000.

For example, when $c=10r$, $a=7r$ and $b=r$, we find that $k_{1,1}-k_{2,2}=6\cdot 4175r$. The true value found above is $6\cdot 4165r$.

Cases in which the Computation is easy.

If $a=l^2h^2-k^2m^2$, $b=l^2k^2-h^2m^2$ and $c=(h^2+k^2)(l^2+m^2)$, where l, h, k and m are any numbers which make a and b positive, then

$$\lambda=2hklm,\ \xi=\frac{lh-km}{lh+km},\ \eta=\frac{lk-hm}{lk+hm}\ \text{and}\ \alpha=\xi\eta.$$

Hence λ, ξ, η and α are rational and this simplifies the calculation.

For example, suppose that $h=k=l=3$ and that $m=1$. Then $\xi=\eta=1/2$, and $\alpha=1/4$. We also have $a=b=72$, $c=180$ and $\lambda=54$.

Hence, by (11),

$$k_{1.2}-k_{2.2}=72+108\left(\frac{1}{3\times 2}+\frac{1}{63\times 8}+\frac{1}{1023\times 32}+\ldots\right)$$
$$=90\cdot 2176.$$

And also by (12),

$$-k_{1.2}=28\cdot 8+108\left(\frac{1}{3\times 4}+\frac{1}{63\times 64}+\ldots\right)$$
$$=37\cdot 8269.$$

Another interesting set of cases is when either of the equations $\omega_1=3\alpha_1$ or $\omega_1=4\alpha_1$ is satisfied. In the former case $a^2=b(b+c)$ and in the latter $c^2b=(a+b)(a-b)^2$. In either case the calculation can be simplified very appreciably.

Systems of Spheres which have the same Mutual Coefficients and the same Orthogonal Sphere.

All the pairs of spheres for which the values of a, b and c satisfy the equations

$$ab\left\{1-\frac{(a-b)^2}{c^2}\right\}=\mathrm{R}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$c-\frac{(a-b)^2}{c}=c_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

have the same orthogonal sphere, the radius of which equals $(c_1{}^2/4-\mathrm{R}^2)^{\frac{1}{2}}$. Every pair of spheres also has the same mutual coefficient as two spheres each of radius R at a distance c_1 apart. The equations (20) and (21) enable us to calculate the mutual coefficient between any two spheres by means of the tables, or by the very simple formulæ given below for two equal spheres. It is also easy to show that the two spheres whose constants are given by

$$\left.\begin{aligned} a&=\frac{c_1}{2}\sqrt{n^2-n}+\left\{\frac{c_1{}^2}{4}(n^2-n)+n\mathrm{R}^2\right\}^{\frac{1}{2}} \\ b&=-\frac{c_1}{2}\sqrt{n^2-n}+\left\{\frac{c_1{}^2}{4}(n^2-n)+n\mathrm{R}^2\right\}^{\frac{1}{2}} \\ \text{and} \qquad c&=nc_1 \end{aligned}\right\}, \quad . \quad . \quad (22)$$

where n is any number not less than unity, have the same mutual coefficient as two equal spheres of radius R at a distance c_1 apart.

The Self-coefficients for Equal Spheres can be computed from the Tables for the Mutual Coefficients.

It is easy to show from Maxwell's formulæ that

$$\frac{k_{1,1}}{\mathrm{R}}=\frac{k_{1,2}}{\mathrm{R}}-\frac{k'_{1,2}}{\mathrm{R}'}\left\{\frac{c}{\mathrm{R}}+2\right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where $c'/\mathrm{R}'=\{c/\mathrm{R}+2\}^{\frac{1}{2}}$. $k'_{1\cdot 2}$ is the mutual coefficient for two equal spheres of radius R' at a distance c' apart. Since c'/R' is known, $k'_{1,2}/\mathrm{R}'$ can be written down from the tables and hence, by (23), $k_{1,1}$ can be found.

Approximate Formulæ for the Capacity Coefficients between Equal Spheres.

When the distance between the spheres is very small the author's formulæ (*l.c. ante*) are the only ones that can be used. For distances between them less than their diameter Kelvin's* and the author's† tables can be used. In the latter tables a small error affects the computed values of the coefficients for the case of x R$=0\cdot45$. In this case the true values are

$$k_{1,1}/\mathrm{R}=1\cdot27173, \qquad -k_{1,2}/\mathrm{R}=0\cdot54728,$$
$$-\tfrac{1}{2}\frac{\partial \mathrm{K}_{1,1}}{\partial x}=0\cdot19961, \qquad \tfrac{1}{2}\frac{\partial \mathrm{K}_{1,2}}{\partial x}=0\cdot23216.$$

For values of c greater than 3R we can use the simple formulæ

$$\frac{k_{1,1}}{\mathrm{R}}=1+\frac{1}{y^2}+\frac{2y^2+1}{(y^3+y)^2-(2y^2+1)^2}-\frac{1}{y^8} \quad . \quad . \quad . \quad (24)$$

and

$$-\frac{k_{1,2}}{\mathrm{R}}=\frac{1}{y}+\frac{y^3+y}{(y^3+y)^2-(2y^2+1)^2}, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where $y=c$ R. When c R$=3$, the maximum inaccuracy of these formulæ is 1 in 10,000. For greater values of c they are much more accurate.

The Capacity between unequal Spherical Electrodes.

It is easy to show that

$$\frac{q}{v_1-v_2}=\frac{k_{1,1}\,k_{2,2}-k^2_{1,2}}{k_{1,1}+k_{2,2}+2k_{1,2}} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

when the charges on the spheres are q and $-q$ respectively and all neighbouring conductors are at zero potential. In this case the ratio of the charge to the difference of potential is constant, and as it is convenient to give it a name the author‡ calls it the capacity between the conductors, when all neighbouring conductors are earthed. By means of the formulæ given above it can be readily computed when there are no other conductors near the spheres. When the spheres are equal, it simplifies to $(k_{1,1}-k_{1,2})/2$. For instance, when $a=b=72$ cms. and $c=180$ cms., then, from the values calculated above, we find that the capacity between the spheres$=64\cdot0222$ cms.$=0\cdot00007114$ microfarad.

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* Reprint, p. 466. Second edition.

† *L.c. ante.*

‡ "Journal" of the Inst. Elec. Eng., Vol. XXX., p. 1022, 1901.

Two spheres, the distance between which can be adjusted by means of a screw, form a very convenient adjustable standard capacity, the values of which in various positions can be written down almost at once to any required accuracy.

The Potentials of the Spheres for Given Charges.

The values of v_1 and v_2 can be easily found from formulæ (1). In the case when the charges are equal and opposite,

$$-\frac{v_1}{v_2}=\frac{k_{2,2}+k_{1,2}}{k_{1,1}+k_{1,2}}. \qquad (27)$$

When the distance between the spheres is infinite

$$-\frac{v_1}{v_2}=\frac{b}{a},$$

and when they are very close together

$$-\frac{v_1}{v_2}=\frac{\psi\{a/(a+b)\}+\gamma}{\psi\{b/(a+b)\}+\gamma}. \qquad (28)$$

For example, when $a=1$ and $b=3$, then at infinity $-v_1/v_2=3$, and when they are almost touching

$$-\frac{v_1}{v_2}=\frac{\psi(1/4)+\gamma}{\psi(3/4)+\gamma}=\frac{6\log 2+\pi}{6\log 2-\pi}=7\cdot178.$$

When the spheres are at the same potential

$$\frac{q_1}{q_2}=\frac{k_{1,1}+k_{1,2}}{k_{2,2}+k_{1,2}},$$

and hence when they are close together

$$\frac{q_1}{q_2}=\frac{\psi\{b/(a+b)\}+\gamma}{\psi\{a/(a+b)\}+\gamma}. \qquad (29)$$

When the spheres whose radii are 1 and 9 respectively, for instance, are at a great distance apart and are at the same potential $q_1/q_2=0\cdot1111$, but when they are touching $q_1/q_2=0\cdot07667$.

Finally, when the charges on the spheres are equal

$$\frac{v_1}{v_2}=\frac{k_{2,2}-k_{1,2}}{k_{1,1}-k_{1,2}}.$$

In this case, when the spheres are a great distance apart $v_1/v_2=b/a$, but when they are very close together $v_1/v_2=1$.

The Capacity Currents to Spherical Electrodes.

If i_1 and i_2 denote the instantaneous values of the currents in the leads to the electrodes, we have

$$\left.\begin{aligned} i_1&=\frac{\partial q_1}{\partial t}=k_{1,1}\frac{\partial v_1}{\partial t}+k_{1,2}\frac{\partial v_2}{\partial t} \\ \text{and} \qquad i_2&=\frac{\partial q_2}{\partial t}=k_{2,2}\frac{\partial v_2}{\partial t}+k_{1,2}\frac{\partial v_1}{\partial t} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence, when the laws according to which the potentials of the terminals vary are known the capacity currents can be found. In practice we generally have either $q_1=-q_2$ or $v_2=0$ at every instant. In the former case,

$$i_1=-i_2=\frac{k_{1,1}k_{2.2}-k^2{}_{2,1}}{k_{1.1}+k_{2\,2}+2k_{1.2}}\frac{\partial}{\partial t}(v_1-v_2, \quad . \quad . \quad (31)$$

and, in the latter,

$$i_1=(k_{1,1}/k_{1,2})i_2=k_{1,1}\frac{\partial v_1}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

ABSTRACT.

Formulæ have been given by Poisson, Kelvin, Kirchhoff, Clerk Maxwell and others for the capacity coefficients of spherical conductors at appreciable distances apart. With the exception of Kirchhoff's formulæ, these formulæ are rarely, if ever, used in practice. The author shows how the remainder after computing a few terms of Kirchhoff's series formulæ can be easily found. The range of their usefulness is therefore considerably extended. This extension makes it possible to simplify very appreciably the author's formulæ for the case, when the spheres are close together. Hence he shows that the numerical computation of the capacity coefficients can be easily made in all cases. Applications showing how the potentials, the capacities and the capacity currents of spherical electrodes can be calculated are given. A simple method of making a variable air condenser whose capacity in every position can be easily calculated to any desired accuracy is also given.

DISCUSSION.

Dr. Eccles pointed out that the readjustment of the terms of a series was not justifiable unless the series was uniformly convergent for the values for which it was to be used. Regarding the proposed standard of capacity made of two opposed spheres, if the capacity be made large, and therefore the distance between the spheres small, errors in the distance become very important. This is in somewhat unfavourable contrast with the concentric cylinder type of condenser, where small errors in setting the axes of the cylinders are not harmful.

The Author pointed out that the series in question was uniformly and absolutely convergent. As a screw is used to adjust the distance between the spheres, the percentage errors need not be greater than the corresponding errors when measuring lengths with a screw gauge. The great advantage of the condenser formed by two spheres is that its capacity can be found at once with high accuracy in all positions. The capacity of a cylindrical condenser can only be accurately calculated when guard cylinders are used. In this case its capacity cannot be varied.

XXXV. *On the Effect of a Narrow Saw-cut in the Edge of a Conducting Strip on the Potential and Stream Lines in the Strip and on the Resistance of the Strip.* By CHARLES H. LEES, *D.Sc., F.R.S., Professor of physics in the East London College, University of London.*

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AT a meeting of the Society some time ago I was asked what would be the effect of a narrow saw-cut in the edge of a standard low-resistance strip, and I have recently put the result of the calculation of the effect into a form suitable for practical purposes.

The conducting strip is taken of breadth unity, of thickness and material such that when unit current flows along it a fall of potential of one unit per unit of length is produced. If the strip is of great length and the saw-cut near the centre of one edge,

$$\frac{dz}{d\zeta}=\mathrm{B}\zeta(\zeta^2-1)^{-1}(\zeta^2-a^2)^{-\frac{1}{2}}$$

where B is a constant, is the Schwarzian equation* for transforming the contour of the strip in the z plane, with the outer end of the cut as origin of co-ordinates, into the axis of the real variable in the ζ plane, the inner end of the cut becoming the origin of co-ordinates, the outer ends the points $\pm a$, where $a<1$, and the ends of the strip the points ± 1 on the real axis in the ζ plane.

Hence
$$z=\mathrm{B}\int\frac{\zeta d\zeta}{(\zeta^2-1)\sqrt{\zeta^2-a^2}}. \quad \ldots\ldots \quad (1)$$

Putting $\sqrt{\zeta^2-a^2}=\sqrt{1-a^2}\tanh\left(\theta\pm i\frac{\pi}{2}\right)$, the $i\frac{\pi}{2}$ being suppressed if $\zeta<1$, we have

$$(\zeta^2-a^2)^{-\frac{1}{2}}\zeta\,.\,d\zeta=\sqrt{1-a^2}\,\mathrm{sech}^2\left(\theta\pm i\frac{\pi}{2}\right)d\theta.$$

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* *See* Werke or Crelle, 70, p. 105; Sir J. J. Thomson "Recent Researches," Chap. III.

and the integral reduces to

$$z=-\frac{B}{\sqrt{1-a^2}}\int d\theta=-\frac{B}{\sqrt{1-a^2}}.\,\theta+\text{constant},$$

that is, $$\sqrt{\frac{\zeta^2-a^2}{1-a^2}}=\tanh\left(-\frac{z\sqrt{1-a^2}}{B}\pm\frac{i\pi}{2}\right),$$

the constant being put$=0$.

Since the breadth of the strip is to be unity $-\frac{\sqrt{1-a^2}}{B}=\frac{\pi}{2}$ and we have finally

$$\sqrt{\frac{\zeta^2-a^2}{1-a^2}}=\tanh\frac{\pi}{2}(z\pm i).\quad.\quad.\quad.\quad.\quad.\quad(2)$$

the i being suppressed if $\zeta<1$.

The depth c of the cut is given by $i\sqrt{\frac{a^2}{1-a^2}}=\tanh\frac{\pi}{2}ic.$

i.e., $$\sqrt{\frac{a^2}{1-a^2}}=\tan\frac{\pi}{2}c,\quad\text{or}\quad a=\sin\frac{\pi}{2}c.$$

If the potential function $w=u+iv$ for any distribution of sources and sinks in the upper half of the ζ plane is known as a function of ζ, it is only necessary to eliminate ζ from the function by means of equation (2) to have the potential function for the corresponding distribution of sources and sinks in the z plane.

For the present purpose we take a source of strength unity at the point $+1$ sending current into the upper half of the ζ plane and a sink of the same strength at the point-1 absorbing this current. The potential function w is then given by the equation

$$w=\frac{1}{\pi}\log\frac{1-\zeta}{1+\zeta},$$

or $\zeta=\tanh\frac{\pi}{2}(w\pm i)$, the i being suppressed if $\zeta<1$. . (3)

Eliminating ζ from the equations (2) and (3) we have

$$\sqrt{1-a^2}\cosh\frac{\pi}{2}(w\pm i)=\cosh\frac{\pi}{2}(z\pm i),$$

that is, $$\cosh\frac{\pi}{2}(w\pm i)=\cosh\frac{\pi}{2}(z\pm i)/\cos\frac{\pi}{2}c,$$

for the potential function w of the strip.

Since w and z will be taken to refer to the same part of the plane we may write the result

$$\cosh\frac{\pi}{2}w=\cosh\frac{\pi}{2}z/\cos\frac{\pi}{2}c. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When the unreal part v of w is zero we have either y_0 the unreal part of $z=0$, and $\cosh\frac{\pi}{2}u_0=\cosh\frac{\pi}{2}x/\cos\frac{\pi}{2}c$, connecting the real parts u_0 and x_0, or $x_0=0$ and $\cosh\frac{\pi}{2}u_0=\cos\frac{\pi}{2}y_0/\cos\frac{\pi}{2}c$, connecting the real part of w with the unreal part of z.

From either equation it follows that the potential at the outer end of the cut$=\frac{2}{\pi}$*arg* cosh sec $\frac{\pi}{2}c$.

When the unreal part v of w is $+1$, y is $+1$, and we have

$$\cosh\frac{\pi}{2}(u_1+i)=\cosh\frac{\pi}{2}(x_1+i)/\cos\frac{\pi}{2}c,$$

or

$$\sinh\frac{\pi}{2}u_1=\sinh\frac{\pi}{2}x_1/\cos\frac{\pi}{2}c,$$

connecting the real parts of w and z.

At a considerable distance, x_0, from the cut along the cut edge we have

$$\sqrt{1-a^2e^{\frac{\pi}{2}u_0}}=e^{\frac{\pi}{2}x_0} \quad \text{or} \quad u_0=x_0+\frac{2}{\pi}\log\sec\frac{\pi}{2}c$$

as a first approximation and

$$\sqrt{1-a^2e^{\frac{\pi}{2}u_0}}(1+e^{-\pi u_0})=e^{\frac{\pi}{2}x_0}(1+e^{-\pi x_0}),$$

or

$$u_0=x_0+\frac{2}{\pi}\left\{\log_\epsilon\sec\frac{\pi}{2}c+\sin^2\frac{\pi}{2}c\,.\,e^{-\pi x_0}\right\} \quad . \quad . \quad . \quad (5)$$

as a second.

The potential difference between a point on the cut edge some distance x_0 from the cut and a point at the bottom of a cut of depth c times the breadth of the strip, is therefore the same as if x_0 were increased by $\frac{2}{\pi}\left\{\log_\epsilon\sec\frac{\pi}{2}c+\sin^2\frac{\pi}{2}c\,.\,e^{-\pi x_0}\right\}$ if the strip is of unit breadth and by b times this expression if the strip is of breadth b.

Similarly at a point on the uncut edge some distance, x_1 from the line of the cut, we have as a first approximation

$$u_1=x_1+\frac{2}{\pi}\log_\epsilon \sec\frac{\pi}{2}c,$$

and as a second

$$u_1=x_1+\frac{2}{\pi}\left\{\log_\epsilon \sec\frac{\pi}{2}c-\sin^2\frac{\pi}{2}c\,.\,e^{-\pi x_1}\right\}. \quad . \quad . \quad (6)$$

The potential difference between a point on the uncut edge some distance x_1 from the line of the cut and the point on the uncut edge opposite to the cut is therefore the same as if the distance to x_1 were increased by $\frac{2}{\pi}\left\{\log_\epsilon \sec\frac{\pi}{2}c-\sin^2\frac{\pi}{2}c\,.\,e^{-\pi x_1}\right\}$ if the strip is of unit breath and by b times this expression if the strip is of breadth b.

The mean of the potential differences between the line of the cut and points on the two edges at equal distances from that line is the same as if the distance of the points from the line of the cut were increased by $\frac{2}{\pi}\log_\epsilon \sec\frac{\pi}{2}c$ if the breadth of the strip is unity and by b times this expression if the breath of the strip is b.

The equations of the equipotential lines and stream lines are determined from equation (4) by expanding and equating separately real and unreal parts. We thus get

$$\left.\begin{array}{ll} & \cosh\frac{\pi}{2}u\,.\,\cos\frac{\pi}{2}v=\cosh\frac{\pi}{2}x\,.\,\cos\frac{\pi}{2}y/\cos\frac{\pi}{2}c \\ \text{and} & \sinh\frac{\pi}{2}u\,.\,\sin\frac{\pi}{2}v=\sinh\frac{\pi}{2}x\,.\,\sin\frac{\pi}{2}y/\cos\frac{\pi}{2}c \end{array}\right\}.$$

Eliminating u and v respectively we have

$$2\cos^2\frac{\pi}{2}c=\frac{(\cosh\pi x+1)(1+\cos\pi y)}{\cosh\pi u+1}+\frac{(\cosh\pi x-1)(1-\cos\pi y)}{\cosh\pi x-1} \quad (7)$$

for the equipotential, and

$$2\cos^2\frac{\pi}{2}c=\frac{(\cosh\pi x+1)(1+\cos\pi y)}{\cos\pi v+1}-\frac{(\cosh\pi x-1)(1-\cos\pi y)}{1-\cos\pi v} \quad (8)$$

for the stream lines.

For numerical calculation of the lines it is better to get equations in x and y separately as follows :—

$$\left.\begin{aligned} 2\sec^2\frac{\pi}{2}c &= \frac{(\cosh \pi u+1)(1+\cos \pi v)}{\cosh \pi x+1}+\frac{(\cosh \pi u-1)(1-\cos \pi v)}{\cosh \pi x-1} \\ &\text{and} \\ 2\sec^2\frac{\pi}{2}c &= \frac{(\cosh \pi u+1)(1+\cos \pi v)}{1+\cos \pi y}-\frac{(\cosh \pi u-1)(1-\cos \pi v)}{1-\cos \pi y} \end{aligned}\right\} \quad (9)$$

By the aid of these equations the equipotential and stream lines for the case in which the cut extends half-way across the strip, that is, $c=\frac{1}{2}$, have been calculated and are shown below along with the equipotential lines found experimentally for a long strip of platinoid.*

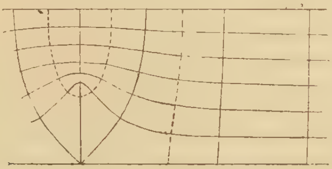

Fig. 1.—Equipotential and Stream Lines in Strip with Cut half across it as calculated.

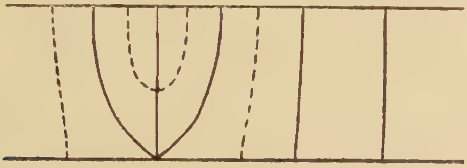

Fig. 2.—Equipotential Lines observed in Long Platinoid Strip.

It will be noticed that the equipotential lines have not become perpendicular to the axis of the strip at a distance from the slit equal to two or three breadths.

In order to illustrate the distortion of the equipotential lines in the neighbourhood of the cut and to give a more definite idea as to the distance from the slit to which it is necessary to go before the lines become normal, the following table has been calculated.

The width of the strip is taken as unity and a current is supposed to pass along it which produces in the uncut strip a fall of potential of unity per unit length. The depth of the cut is c.

---

* For the numerical calculation of these curves and for the experimental verification I have to thank Mr. T. E. G. McCathie, B.Sc.

Table of Positions of Points at Given Potentials on Edges of Strip of Breadth Unity.

Potential.	$c=\frac{1}{4}$.		$c=\frac{1}{2}$.		$c=\frac{3}{4}$.	
	Distances.		Distances.		Distances.	
	Uncut.	Cut.	Uncut.	Cut.	Uncut.	Cut.
0	0	0	0	0	0	0
$1 \cdot \frac{2}{\pi}$	$0{\cdot}941\frac{2}{\pi}$	$0{\cdot}893\frac{2}{\pi}$	$0{\cdot}758\frac{2}{\pi}$	$0{\cdot}424\frac{2}{\pi}$	$0{\cdot}436\frac{2}{\pi}$	0
$2 \cdot \frac{2}{\pi}$	1·924 ,,	1·918 ,,	1·671 ,,	1·634 ,,	1·131 ,,	$0{\cdot}906\frac{2}{\pi}$
$3 \cdot \frac{2}{\pi}$	2·921 ,,	2·920 ,,	2·656 ,,	2·651 ,,	2·054 ,,	2·025 ,,
$4 \cdot \frac{2}{\pi}$	3·921 ,,	3·921 ,,	3·654 ,,	3·653 ,,	3·041 ,,	3·037 ,,
$5 \cdot \frac{2}{\pi}$	4·921 ,,	4·921 ,,	4·653 ,,	4·653 ,,	4·040 ,,	4·039 ..
$6 \cdot \frac{2}{\pi}$	5·921 ,,	5·921 ,,	5·653 ,,	5·653 ,,	5·039 ,,	5·039 ,,

From this table it will be seen that in the case of the deep cut $c=\frac{3}{4}$ it is necessary to go to a distance from the slit equal to three times the breadth of the strip before the equipotential lines become practically parallel and at unit distance apart.

Since the effect of a fine saw-cut in and perpendicular to the edge of a long resistance strip and of depth c times the breadth of the strip, is equivalent to an increase of length of the strip equal to $\frac{4}{\pi} \log_e \sec \frac{\pi}{2} c$ breadths, the depth of cut necessary to raise the resistance of the strip by any given amount may be calculated.

ABSTRACT.

By means of the Schwarzian transformation it is shown that if a fine saw-cut is made in the edge of a long conducting strip, perpendicular to the edge, the resistance of the strip is increased by an amount equivalent to that of a length of strip equal to $\frac{4}{\pi} \log_e \sec. \frac{2}{\pi} c$. breadths, where c is the quotient of the depth of the cut by the breadth of the strip. Diagrams of the stream and potential lines near the cut are given.

XXXVI. *The Probability Distribution of the Time Intervals of α Particles with Application to the Number of α Particles emitted by Uranium.* By E. MARSDEN, *M.Sc., and* T. BARRATT, *B.Sc., East London College, University of London.*

RECEIVED JULY 11, 1911.

IN counting the α particles from radio-active substances by the scintillation method the great preponderance of short intervals is very noticeable, especially when the scintillations are appearing at a slow rate. In fact this preponderance leads one at first sight to consider the α particles as coming in groups and not distributed according to a simple probability law. Rutherford and Geiger* have recently examined the emission of α particles from polonium from this point of view. In their experiments scintillations produced on a definite area of a zinc-sulphide screen placed in an exhausted vessel at a distance from the polonium source of α particles, were recorded on a travelling chronograph tape and analysed according to the numbers occurring in successive equal time intervals, generally large compared with the average time interval of the scintillations. The results were found to be in good agreement with Bateman's† theoretical formula that $x^n e^{-x}/n$ is the probability of n scintillations occurring in a given interval where x is the true average number for the interval. However, were the particles given off in equal groups and the groups distributed in time according to probability it is conceivable that the above formula would still hold. It would, therefore, seem preferable in many respects to test the application of the probability laws to actual time intervals between successive α particles and to count the whole number from a given source instead of only those emitted within a relatively small solid angle.

The problem of the calculation of the distribution of time intervals can be approached in the following way. Let the average time interval between successive α particles falling on a zinc-sulphide screen from a radio-active source be $1/\mu$, and let us assume that the time under consideration is small compared with the time period of the radio-active substance. Assuming that at time 0 the observer sees a scintillation, let us find the probability that a time interval, t, elapses without the occurrence of a second scintillation but that within a further very small interval, δt, a scintillation occurs. This problem corresponds exactly to the method employed in practice. The probability that no scintillation occurs in the time interval t is $e^{-\mu t}$.* The probability that a scintillation occurs in the time interval from t to $t+\delta t$ is independent of t and is $\mu\delta t$. As these events are independent of one another the probability of an interval from t to $t+\delta t$ is the product of their separate probabilities and is therefore $\mu e^{-\mu t}\delta t$; or in a large number of N intervals the probable number of intervals larger than t and smaller than $t+\delta t$ is $N\mu e^{-\mu t}\delta t$.

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* Rutherford and Geiger, "Phil. Mag." XX., p. 698, Oct., 1910.

† *Loc. cit.* Note by H. Bateman.

This result is at first sight somewhat surprising,† for it indicates that whatever the value of μ small intervals are more probable than large ones, whereas one would at first sight expect that the intervals would be distributed according to a law somewhat similar to that of Maxwell for the distribution of the velocities of the molecules of a gas.

The above formula has been applied to several sets of observations made by us on the α particles from polonium and uranium and found to agree well with experiment. The case of uranium is particularly interesting, for Boltwood‡ and Geiger and Rutherford§ have shown that uranium emits two α particles. If these α particles are given off together or by successive α ray products, the second having a period less than a few seconds, then in observing scintillations caused by α particles from uranium placed in contact with a zinc-sulphide screen one would expect a larger number of short intervals than according to the simple theory above. The experimental arrangement used to test this point is shown in Fig. 1 and is the same as that of Geiger and Marsden‖ in their experiments on the α particles from actinium and thorium emanations. S_1 and S_2 are two screens of zinc-sulphide coated on glass with the zinc-sulphide coatings facing each other and separated by sheets of aluminium foil. One of these sheets was coated with a very fine layer of uranium-oxide produced by pouring on it uranium-oxide suspended in alcohol and allowing the alcohol to evaporate. Two microscopes, M_1 and M_2, were focussed on exactly opposite areas of the two screens, and the scintillations observed were recorded by two separate observers on the same travelling tape of a Morse inker by means of two needle points arranged to puncture the tape by arrangements of levers. Ink marks were also made on the tape at second intervals by means of a circuit containing an electromagnet and a pendulum contact. Just sufficient sheets of thin aluminium foil were used that no scintillations on one screen were visible in the microscope focussed on the other, more than one being necessary on account of the difficulty of keeping it unpunctured by the crystals of the screen on subsequently pressing them together. When the screens were placed together in this way they were found to be about $\frac{1}{3}$ mm. apart, while the diameters of the fields of the two similar microscopes were about 2·3 mm., so that practically the whole of the particles given off by the uranium between the screens would impinge on either one or the other of them. In making the observations the tape was started and scintillations recorded by both observers for a period of from two to five minutes. The tape was then stopped and after the eyes had been rested another set of observations was made, and so on, until a sufficient number of scintillations had been observed. The records on the tape were then examined and a curve drawn giving the number of occurrences of different intervals, and this curve compared with that obtained from the formula given above. Thus in the particular case shown in Fig. 2.

---

* H. Bateman, *loc. cit.* It is not at first sight clear that we are justified in applying this formula to a time interval commencing with a scintillation, but the time of occurrence of the scintillation is really arbitrary as referred to the times of previous scintillations. The result may also be obtained as follows: The probability that a time interval that does not contain a scintillation is $e^{-\mu t}$, and this is, therefore, the probability that an interval without scintillation is greater than t, or in N such intervals the number greater than t is $Ne^{-\mu t}$. Similarly, the number of intervals which are greater than $t+\delta t$ is $Ne^{-\mu(t+\delta t)}$. The difference gives us the number of intervals between successive scintillations greater than t but smaller than $t+\delta t$, and is $N\mu e^{-\mu t}\delta t$.

† *Cf.* Rutherford and Geiger, Roy. Soc. "Proc.," A. LXXI., p. 141 (1908).

‡ Boltwood, "Amer. Journ. Sci.," Vol. XXV., p. 270 (1908).

§ Geiger and Rutherford, "Phil. Mag.," XX., p. 692, 1910.

‖ Geiger and Marsden, "Phys. Zeit.," XII., p. 7, 1910.

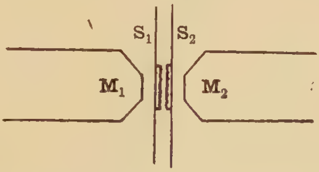

FIG. 1.

Total time$=$2,988·5 seconds.

Number of intervals$=$(taking the observations on both screens) 319.

Average interval$=$9·37 seconds. $\mu=0{\cdot}1068$ (approx.).

The probability of an interval from 0 to 1 second is, to the approximation sufficient for the present experiment, the value of $N\mu e^{-\mu t}$ for $t=0{\cdot}5=319\times 0{\cdot}1068\, e^{-0{\cdot}0534}=32{\cdot}2$.

The actual number of such observations was 36.

In this way the probability curve shown in Fig. 2 has been drawn and it will be seen that the actual values observed, which are indicated by the points, agree with it to an extent within the limits of the probability error. The tape was also examined by the method of Rutherford and Geiger, the number of scintillations in successive intervals of 16 seconds being measured. The number of occurrences of each number are given in column I. of the following table, while in column II. are the numbers calculated from Bateman's formula.

No. of particles in interval of 16 seconds.	No. of groups.	
	I. Experimental.	II. Calculated.
0	35	34
1	56	58
2	49	49·5
3	32	28
4	9	12
5	5	4·1
6	1	1·15

The agreement is good, and considering the small number of observations it is probably fortuitous. Good agreement between experiment and theory was also found in other cases in which the amount of uranium was varied so as to give different rates of emission of α particles, while experiments in which a layer of polonium was used also showed excellent agreement with theory. The polonium was obtained by evaporating on an aluminium foil a solution of the active deposit on the walls of an old tube which had contained radium emanation. The emission of α particles was also examined by the foregoing methods on two separate screens placed at various distances up to 83 cm. from sources of polonium. Agreement between experiment and theory was in all cases satisfactory.

Through the kindness of Prof. Rutherford and Dr. Geiger, we have been allowed to examine some of the records of the scintillations made in their experiments on the "Probability Variations in the Distribution of α Particles." These also showed excellent agreement with the above theory.

The theory of these observations is independent of whether the screens are continuous or whether scintillations are only produced by a fraction of the α particles. However (with the screens used) it is unlikely that more than 10 per cent. of the α particles failed to produce scintillations. The chief source of error might be the missing of scintillations through temporary fatigue of the eyes, for in the case of uranium the range of the particles is only 2·7 cm. and their velocity consequently small, this being further cut down by the aluminium foils. Using the low magnification necessary to obtain a large field

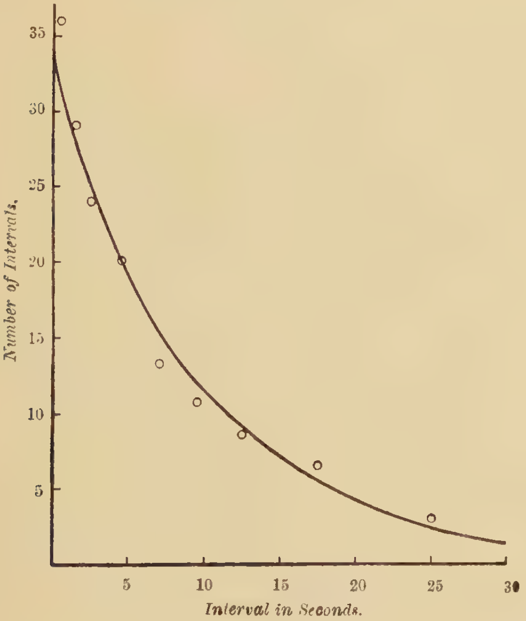

FIG. 2.

and, with the particular objectives used, the consequent small solid angle of the light from the scintillations, they were much fainter than in usual scintillation experiments. The missing of scintillations, however, is partly avoided by the observations of one observer forming a check on those of the other. To test this point further and to verify the experimental arrangement the observations of Geiger and Marsden* on the double scintillations from actinium emanation were repeated. The emanation was allowed to diffuse up between the two screens and the same numbers of connected scintillations were observed as in their experiments. Thus, with the emanation and active deposit in equilibrium nearly 50 per cent. of the scintillations appeared simultaneously on either the same or both screens, and a simple calculation showed that with the arrangement used only a few of the α particles could have been missed. As a further support of the theory it was found that when the doubles with actinium emanation were neglected and counted as single scintillations the distribution of the remaining time intervals was in agreement with theory.

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* *Loc. cit.*

We thus see that the variation of the intervals between the emission of successive α particles from both polonium and uranium shows good agreement with simple probability theory, and we therefore conclude that the two α particles from uranium are not given off simultaneously. Further, if they are given off by successive α-ray products the period of the second must be greater than a few seconds. This is also unlikely because of the small ranges of both sets of α particles.*

Further experiments are in progress to apply the foregoing method to the α particles from thorium B and thorium C and also to investigate the possible existence of a short-life product in the active deposit of actinium indicated by the 10 per cent. of connected particles found in the experiments of Geiger and Marsden.

We are indebted to Prof. Lees, F.R.S., for his kind interest in these experiments.

* * *

ADDENDUM.

[JULY 18, 1911.]

Prof. Rutherford and Dr. Geiger have very kindly allowed us to publish the figures relating to observations on some of the records of the scintillations obtained in their experiments. The time intervals between successive scintillations were deduced from the measured distance between the records on the tapes, allowance being made for the varying velocity of the tape by reference to the time marks. The time marks were made at 30 seconds intervals and the average distance between them was about 7 cm.

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* A. Foch, "Le Radium," VIII., p. 101, 1911.

The results are given in the following table. Column II. gives the experimentally determined number of intervals of duration between the limits given in Column I. The theoretical numbers in Column III. were calculated as follows :—

Total number of intervals............	= 7,564
Total time	=14,598 secs.
Average interval....................	= 1·930 secs.

$$\mu = 0{\cdot}5181.$$

Number of intervals of duration greater than t_1 and less than

$$t_2 = \int_{t_1}^{t_2} \mathrm{N}\mu e^{-\mu t}dt = \mathrm{N}(e^{-\mu t_1} - e^{-\mu t_2}) = 7{,}564(e^{-0{\cdot}5181 t_1} - e^{-0{\cdot}5181 t_2}).$$

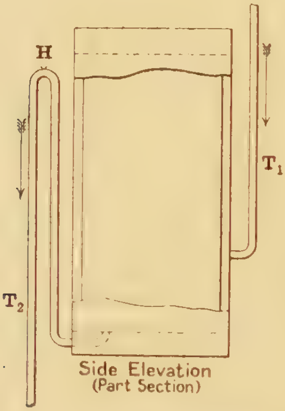
Duration of interval between successive scintillations. Seconds.	Number of intervals.	
	Experimental.	Calculated.
0·0 to 1·0	3,106	3,059
1·0 to 2·0	1,763	1,822
2·0 to 3·0	1,115	1,085
3·0 to 4·0	658	646
4·0 to 5·0	389	385
5·0 to 6·0	206	229
6·0 to 7·0	130	136
7·0 to 8·0	86	81
8·0 to 9·0	42	48
9·0 to infinity	68	71

The agreement between the experimental and theoretical values is surprisingly good and is probably well within the probability error.

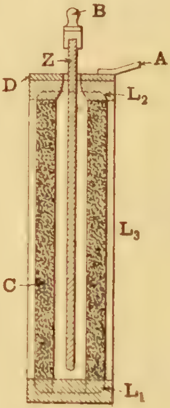
XXXVII. *On the Benkö Primary Battery.* *By* W. R. Cooper, *M.A., B.Sc.*

June 30, 1911.

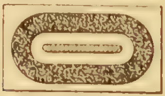
The Benkö primary battery, due to Mr. Stephan Benkö, of Buda-Pest, depends essentially upon the use of a porous carbon electrode through which the electrolyte flows. The construction is shown diagrammatically in the adjoining figure. Here the carbon, which may be described as a flattened cylinder open at both ends, is indicated at C. This is closed at the bottom by a lead cap, L_1, and the top is finished off by a lead ring L_2, these being put on under pressure so that intimate contact is made with the carbon. A lead shell, L_3, is then fitted round the carbon, leaving a small space between the carbon and the shell, and is jointed to the lead cap and lead ring, thus forming a chamber round the carbon. One terminal is fitted to the lead L_2 and the other is formed by the zinc plate Z, which is held in an ebonite frame. The electrolyte is introduced under a suitable head into the lead shell by a tube, T_1; it flows through the carbon to the zinc and



Side Elevation
(Part Section)



Cross Section



Plan
(Section)

thence to waste through an inverted U tube T_2, so that the electrolyte is maintained at a proper level. Syphoning is prevented by a small hole at H.

The electrolyte (a sodium bichromate solution) is supplied at such a head as to give a flow of ½ litre, or more, per hour per cell, depending upon the current required. The E.M.F. is 2 volts.

Polarisation is thus reduced to a minimum and remains constant for a given current, provided the cell is not overloaded. The size of cell referred to is 9¼ in. high overall, 6¼ in. long and 1¾ in. wide, weighs about 10 lb. excluding the tank for electrolyte, and will give a steady current (depending upon the strength of the electrolyte and rate of flow) up to, say, 25 amperes at 1·5 volts. The internal resistance is below 0·01 ohm. There is no "mess" and no difficulty in keeping in order. The cell is, therefore, very suitable for laboratory use, or work where the cost is not of prime importance. Various types are being made according to the work required, and include cells which depend for their action upon diffusion as distinct from a steady flow.

DISCUSSION.

Mr. G. L. Addenbrooke asked whether the contact between the lead and the carbon remained satisfactory over a long time.

The Author replied in the affirmative so far as his experience (extending over 15 months) had gone.

The President inquired as to the maximum number of watts that could be taken from the battery continuously, and whether, weight for weight, more power could be taken from it than from accumulators.

The Author replied to the first part of the question about 25 to 40 watts, depending on the solution, and stated that, excluding the reservoir, 2·4 to 4 watts per pound could be taken from the battery as compared with ½ watt (8-hour rate) up to 2½ watts (automobile cells at 5-hour rate) for accumulators.

Replying to Dr. Willows the author said that the zinc could be taken out, or the electrolyte could be run out when the cell was not in use.

Mr. W. Duddell asked how the battery would compare with accumulators for giving a steady current of the order of 5 amperes to 1 part in 5,000 or so. Accumulators could only be relied on to keep steady to about 1 part in 1,000.

The Author said he could not reply definitely, but he would expect a better result from the Benkö battery bulk for bulk. It would be necessary to maintain a constant temperature.

Mr. Frazer inquired how long the diffusion cell lasted before it became an ordinary cell.

The Author replied that the E.M.F. began to fall after about 10 hours in use, and the diffusion would probably be complete for practical purposes in about five days in the small cells if not in use.

Mr. S. W. Melsom asked whether the electrochemical efficiency of the zinc was the same as in an ordinary primary cell.

The Author stated that it was about the same.

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